Solve all 5 problems. Write your solutions only where indicated, or write explicitly, "continued on page k".
Neatness counts! Language counts!

Problem 1. Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be a pair of continuous functions.

1. Show that $\{x \in X\colon f(x) = g(x)\}$ is a closed set in X.
2. A subset $D \subset X$ is called *dense* if it has a non-empty intersection with every ~~closed~~ non-empty open set in X (example: $D = \mathbb{Q}$ is dense in $X = \mathbb{R}$). With f and g as before, show that if f is equal to g on a dense set D, then f is equal to g everywhere.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Your solution of Problem 1.

1. We instead show that $\{x \in X: f(x) \neq g(x)\}$ is open. Let $A, B \subseteq Y$ such that A, B are open in the topology of Y & $A \cap B = \emptyset$. Then

$$f^{-1}(A) \cap g^{-1}(B) = \{x \in X : f(x) \in A \wedge g(x) \in B\}$$

is open in X, since f & g are cont, & contains only points which satisfy $f(x) \neq g(x)$, since A & B are disjoint. Thus, our set is

$$\{x \in X : f(x) \neq g(x)\} = \bigcup_{\substack{A, B \in T_Y \\ A \cap B = \emptyset}} f^{-1}(A) \cap g^{-1}(B)$$

& is open as desired so that $\{x \in X : f(x) = g(x)\}$ is closed.

2. We prove by contradiction: suppose that set $\{x \in X : f(x) \neq g(x)\}$ is nonempty. By Part 1, this set is open, since it is the complement of $\{x \in X : f(x) = g(x)\}$; thus, since D is dense,

$$D \cap \{x \in X : f(x) \neq g(x)\} \neq \emptyset.$$

However, for all $x \in D$, $f(x) = g(x)$, so we can't have $f(x) \neq g(x)$ as well; thus, $f \equiv g$ as desired. ∎

Problem 2. Show that if a topological space X is Hausdorff and if $x_1, \ldots, x_n$ are distinct points in X, then there exist open sets $U_1, \ldots, U_n$ in X such that $\forall i\, x_i \in U_i$ and such that $\forall i \neq j$, $U_i \cap U_j = \emptyset$.

Tip. In math exams, "show" means "prove".

Your solution of Problem 2.

We proceed via induction. For our base case ($n=1$), there is nothing to check. Now suppose that for $n=k$, there exists open sets $V_1, \ldots, V_k$ in X s.t. $x_i \in U_i$ for all $i \in \{1, \ldots, k\}$ & s.t. $V_i \cap V_j = \emptyset$ if $i \neq j$. We wish to construct this for $n=k+1$. Since X is Hausdorff, we know we can find neighbourhoods $W_1, \ldots, W_k$ of x_{k+1} & neighbourhoods Z_i of x_i resp. s.t. $W_i \cap Z_i = \emptyset$. Then let

$$U_1 = V_1 \cap Z_1$$

$$\vdots$$

$$U_k = V_k \cap Z_k$$

$$U_{k+1} = \bigcap_{j=1}^{k} W_j.$$

We claim these are the sets we need. Note that all of these are neighbourhoods of their respective x_i's by construction; furthermore, they are disjoint: if $i \neq k+1$ & $j \neq k+1$, $U_i \cap U_j = V_i \cap V_j = \emptyset$ by hypothesis & if (without loss of generality) $i = k+1$, then W_j & Z_j are disjoint by construction, so too must U_j & U_{k+1}. This completes the induction.

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Problem 3. Let X be a metrizable space and let $A \subset X$ be some subset of X. Show that a point x belongs to $\bar{A}$ if and only if there exists a sequence a_i of points in A such that $a_i \to x$.

Tip. "If and only if" always means that there are two things to prove.

Your solution of Problem 3.

Suppose first we have a sequence $\{a_i\}$ of points in A that converge to x. We will show that $x \in \bar{A}$. It is sufficient to show that for any ball B centered at x w/ radius ε, $B \cap A$ contains at least one of the points of $\{a_i\}$; however, by the definition of a convergent sequence in a metric space, we know that after some $N \in \mathbb{N}$, $d(a_n, x) < \varepsilon$ for all $n \geq N$. Thus, every basic neighborhood of x has a nonempty intersection w/ A & $x \in \bar{A}$.

Now suppose that $x \in \bar{A}$; we wish to construct a sequence $\{a_i\}$ of points in A s.t. $a_i \to x$. To do this, consider balls centered at x w/ radius $1/n$ which we will denote $B_{1/n}(x)$. Note that since $x \in \bar{A}$, $A \cap B_{1/n}(x) \neq \emptyset$, so for each $n \in \mathbb{N}$ we can choose a point $a_n \in A \cap B_{1/n}(x)$. Clearly this is a sequence of points in A, so if we can show they converge to x that completes the proof. We do this via the ε-definition of convergence: given $\varepsilon > 0$, choose $N = \lceil 1/\varepsilon \rceil$. Then for all $n \geq N$,

$$d(a_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon \quad \text{as desired.} \quad \blacksquare$$

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Problem 4. Let X_n be a sequence of non-empty topological spaces whose topology is not the trivial topology. Show that the boxes topology on $X := \prod_{n=1}^{\infty} X_n$ is strictly stronger than the cylinders topology on X.

Tip. Here and almost always, a concise yet precise solution is better than a lengthy roundabout one.

Your solution of Problem 4.

Since any cylinder is a box, it is clear that the boxes topology contains the cylinders topology. We wish to construct a set which is open in the box topology but is not open in the cylinder topology. To do this, for each $i \in \mathbb{N}$, pick $U_i \subseteq X_i$ s.t. $U_i \neq \emptyset, U_i \neq X_i$ & U_i open (which we can do since the topologies given are nontrivial). We claim

$$U := \prod_{i=1}^{\infty} U_i$$

is open in the boxes topology but not in the cylinders. As U is the product of open sets, it is open in the boxes topology; however, we will show that no open neighbourhood of a point $x \in U$ can be contained in U under the cylinder topology. It is sufficient to consider basic neighbourhoods of x, i.e. of the form

$$V := \prod_{i=1}^{\infty} V_i \quad \text{where finitely many } V_i \ni \pi_i(x) \text{ are not } \subsetneq X_i.$$

If we constrain a finite number of V_i's, there must exist an $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $V_n = X_n$. But then since $U_n \subsetneq X_n$, we cannot have $V_n \subseteq U_n$, & so our box is not open in the cylinders topology. ∎

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Problem 5. Let X be a topological space, let $\sim$ be an equivalence relation on X, let $Y = X/\sim$ be the quotient set, and let $\pi\colon X \to Y$ be the natural projection.

1. Write the "functional" definition of the quotient topology on Y. Namely, list some functions or some families of functions whose range is Y or whose domain is Y that must be continuous relative to the quotient topology on Y.
2. Prove that if a quotient topology on Y exists, then it is unique. (Note that you are not asked to prove that the quotient topology exists!).

Tip. For all problems, you may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space allocated for the solutions.

Tip. Once you have finished writing an exam, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.

Your solution of Problem 5.

1. We wish the quotient topology to satisfy the following:
(a) $\pi: X \to Y$ (natural projection) is cont.
(b) given $f: Y \to Z$, if $f \circ \pi$ is cont, so should f be cont.

2. Suppose $\mathcal{T}$ & $\mathcal{T}'$ are both topologies on Y satisfying (a) & (b). We can illustrate their relationship by the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (Y, \mathcal{T}) \arrow[rr, shift left, "id_1"] & & (Y, \mathcal{T}') \arrow[ll, shift left, "id_2"] \end{tikzcd}$$

It is sufficient to show that id_1 & id_2 are cont. to prove that our two topological spaces are equivalent. For id_1, note that $id_1 \circ \pi_1 = \pi_2$. Since $\mathcal{T}'$ satisfies (a), π_2 is cont; but since $\mathcal{T}$ satisfies (b), id_1 is cont. We can argue exactly similarly for id_2, so that both identity maps are cont. & $\mathcal{T} = \mathcal{T}'$ (i.e. the quotient topology is unique) as desired.