MAT327 – HW10

2024-12-10

Q1

Complete the definition of $\pi_1 : \mathbf{Top}_0 \to \mathbf{Grp}$ and prove that it is a functor.

Answer.

Claim. *Let $f : (X, x_0) \to (Y, y_0)$ be a homeomorphism. Then, the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$ is well-defined and a group homomorphism.*

Proof. By HW8 Q2b, if γ_0 is path-homotopic to γ_1, then $f \circ \gamma_0$ is homotopic to $f \circ \gamma_1$, so the map f_* is well-defined. Moreover, we have

$$\begin{aligned} f_*([\alpha] * [\beta]) &= [f \circ (\alpha * \beta)] \\ &= [(f \circ \alpha) * (f \circ \beta)] \\ &= f_*([\alpha]) * f_*([\beta]) \end{aligned}$$

where the second equality follows by inspection of the definition of $*$. Thus, f_* is a group homomorphism. ■

Define π_1 by mapping the based-space (X, x_0) to the fundamental group $\pi_1(X, x_0)$ and mapping morphisms $f : (X, x_0) \to (Y, y_0)$ to the morphism f_*.

By our claim above, the map f_* is well-defined and a group homomorphism, so π_1 is a well-defined functor.

We show that π_1 is functorial in that it distributes over composition and preserves identities. Indeed, for every based-space (X, x_0), we have

$$\pi_1\left(\mathrm{id}_{(X,x_0)}\right)([\gamma]) = \left[\mathrm{id}_{(X,x_0)} \circ \gamma\right] = [\gamma]$$

so π_1 maps identities to identities. Also, given based-spaces $(X, x_0), (Y, y_0), (Z, z_0)$ and morphisms $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, we have

$$\begin{aligned} \pi_1(g \circ f)([\gamma]) &= [(g \circ f) \circ \gamma] \\ &= [g \circ (f \circ \gamma)] \\ &= \pi_1(g)([f \circ \gamma]) \\ &= \pi_1(g)(\pi_1(f)(\gamma)) \\ &= (\pi_1(g) \circ \pi_1(f))([\gamma]) \end{aligned}$$

so π_1 distributes over composition.

Thus, $\pi_1 : \mathbf{Top}_0 \to \mathbf{Grp}$ is a functor of categories. ■