MAT327 – HW6

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Q2

Let $f : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{R}$ be a continuous function. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$.

Answer.

Let $g : S^1 \to \mathbb{R}$ be defined by $g(z) = f(z) - f(-z)$.

Fix $z_0 \in S^1$ and consider $a \coloneqq g(z_0)$. Then,

$$-a = f(-z) - f(z) = f(-z) - f(-(-z)) = g(-z_0)$$

Since S^1 is connected and g is continuous, we have $g(S^1)$ connected. The above shows that both $a, -a \in g(S^1)$, and since the only connected subspaces of $\mathbb{R}$ are convex, we thus have (assuming without loss of generality that $a \geq 0$), that $[-a, a] \subset g(S^1)$. But then, $-a \leq 0 \leq a$, so $0 \in g(S^1)$, which is to say there exists some $z \in S^1$ such that

$$0 = g(z) = f(z) - f(-z)$$

and hence $f(z) = f(-z)$. ■

Q3

If $A \subset X$ and A is path connected, is $\overline{A}$ always path connected too?

Answer. **No**. The topologist's sine curve yields a counter example.

Set $X = \mathbb{R}^2$ and take $A \subset X$ to be

$$A = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\}$$

Then, A is path connected because given any $\left(t_0, \sin\left(\frac{1}{t_0}\right)\right), \left(t_1, \sin\left(\frac{1}{t_1}\right)\right) \in A$, the function $\alpha : [0,1] \to A$ defined by

$$\alpha(t) = \left(t_0 + t(t_1 - t_0), \sin\left(\frac{1}{t_0 + t(t_1 - t_0)}\right) \right)$$

is a path from t_0 to t_1 lying in A.

We will show that $\overline{A}$ is not path connected. First, a technical lemma.

Lemma. *Let* $r > 0$ *and* $y \in [-1, 1]$*. There exists* $0 < t < r$ *such that* $\sin\left(\frac{1}{t}\right) = r$.

Proof. Since sin is surjective, we can choose some $t' \geq 0$ such that $\sin(t') = y$. Then, choose $N \in \mathbb{N}$ large enough so that $t = \left|\frac{1}{t' + 2\pi N}\right| < r$, so that

$$\sin\left(\frac{1}{t}\right) = \sin(t' + 2\pi N) = \sin(t') = y$$

■

Claim. $\overline{A} = A \cup (\{0\} \times [-1, 1])$.

Proof. We show the $(\supset)$ inclusion first.

Let $(x, y) \in A \cup (\{0\} \times [-1, 1])$. If $(x, y) \in A$, we're done. Otherwise, $x = 0$ and $y \in [-1, 1]$. Given any neighborhood $B_r((x, y))$ of (x, y), we can choose by our lemma some $0 < t < r$ such that $\sin\left(\frac{1}{t}\right) = y$, so that $\left(t, \sin\left(\frac{1}{t}\right)\right) \in A$ and $\left|(x, y) - \left(t, \sin\left(\frac{1}{t}\right)\right)\right| = |t| < r$. In other words, $B_r((x, y))$ intersects A. So, $(x, y) \in \overline{A}$.

For the other inclusion, let $(x, y) \in \overline{A}$. Since $A \subset [0, \infty) \times [-1, 1]$ which is closed, we must have $\overline{A} \subset [0, \infty) \times [-1, 1]$, so $x \geq 0$ and $y \in [-1, 1]$. If $x = 0$, then $(x, y) \in \{0\} \times [-1, 1]$ and we're done. Otherwise, $x > 0$. We'll show that $y = \sin\left(\frac{1}{x}\right)$ by showing $\left|y - \sin\left(\frac{1}{x}\right)\right| < \varepsilon$ for every $\varepsilon > 0$. Given $\varepsilon > 0$, choose by continuity some $\delta > 0$ such that if $|x - x'| < \delta$, then $\left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x'}\right)\right| < \frac{\varepsilon}{2}$. Then, there exists a neighborhood U of (x, y) sufficiently small so that every $(x', y') \in U$ satisfies $|x - x'| < \delta$ and $|y' - y| < \frac{\varepsilon}{2}$. Since $(x, y) \in \overline{A}$, we can find $(x', y') \in U \cap A$, so that $y' = \sin\left(\frac{1}{x'}\right)$, and hence

$$\left|y - \sin\left(\frac{1}{x}\right)\right| \leq \left|y - \sin\left(\frac{1}{x'}\right)\right| + \left|\sin\left(\frac{1}{x'}\right) - \sin\left(\frac{1}{x}\right)\right| < \varepsilon$$

So, $y = \sin\left(\frac{1}{x}\right)$ and hence $(x, y) \in A \subset A \cup (\{0\} \times [-1, 1])$.

Both inclusions show $\overline{A} = A \cup (\{0\} \times [-1, 1])$ as needed. ■

Finally, we show that $\overline{A}$ is not path connected. The following proof was adapted from Munkres.

Suppose for the sake of contradiction that $\overline{A}$ is path connected, so that there exists a path $\gamma' : [0,1] \to \overline{A}$ from $(0,0)$ to some point $(x_0, y_0) \in A$.

The preimage $\gamma'^{-1}(\{0\} \times [-1,1])$ is closed by continuity, so it has a largest element b and so, since $\overline{A} = A \cup (\{0\} \times [-1,1])$, the image $\gamma'((b,1])$ is contained in A. By translating and scaling as necessary, we obtain a continuous map $\gamma : [0,1] \to \overline{A}$ where $\gamma(0) = b \in \{0\} \times [-1,1]$ and $\gamma(t) \in A$ for $t > 0$, which is to say $\gamma(t) = (x(t), y(t))$ for some continuous $x, y : [0,1] \to \mathbb{R}$ where $x(t) = 0$ and $y(t) = \frac{1}{\sin(x(t))}$ for $t > 0$.

We define sequences $(u_n), (t_n)$ as follows: for each n, choose by our lemma some some $x(0) = 0 < u_n < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right) = (-1)^n$ and apply the intermediate value theorem to x to obtain some $0 < t_n < \frac{1}{n}$ such that $x(t_n) = u_n$.

Then, the sequence (t_n) converges to 0, but the sequence $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = \sin\left(\frac{1}{u_n}\right) = (-1)^n$ does not converge, contradicting continuity of y.

Thus, $A \subset \mathbb{R}^2$ is a path connected set whose closure is not path connected, so it is not true that the closure of path connected sets is always path connected. ∎

Q4

Show that an open connected subset U of $\mathbb{R}^n$ is path-connected.

Answer.

Recall that we defined connected-ness to exclude empty sets, so $U \neq \emptyset$ and fix $x_0 \in U$. Define

$$S = \{x \in U : \text{there exists a path in } U \text{ from } x_0 \text{ to } x\}$$

We claim that S is both open and closed in U.

For S open, let $x \in S$ be arbitrary. Since $x \in U$, and U is open, there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(x) \subset U$. We'll show that every $x' \in B_\varepsilon(x)$ is also in S. Fix $x' \in B_\varepsilon(x)$. Since balls in $\mathbb{R}^n$ are convex, the straight line path connecting x and x' lies in $B_\varepsilon(x)$ and therefore in U. Since x_0 is connected to x via a path in U and x is connected to x' via a path in U, we have a path from x to x' lying in U, so that $x' \in S$. Thus, S is open.

Now, for S closed, we use the equivalence of closure and sequential closure in metric spaces. Let $x \in \overline{S}$ (where $\overline{S}$ is the closure in U), so that there exists a sequence (x_n) in S converging to x. Since $x \in U$, we have some $\varepsilon > 0$ such $B_\varepsilon(x) \subset U$, and since $x_n \longrightarrow x$, we have some $N \in \mathbb{N}$ such that $x_N \in B_\varepsilon(x)$. Again, balls are convex, so the straight line path connecting x_N and x lies in $B_\varepsilon(x)$ and therefore in U, and so, since x_0 is connected to x_N via a path in U and x_N is connected to x via a path in U, we have a path from x_0 to x via a path in U. Thus, S contains its closure and hence is closed.

Now, the constant path gives a path from x_0 to itself, so certainly $x_0 \in S$, and hence S is not-empty. But, given that U is connected, the only non-empty, clopen subset of U is U itself, so that $S = U$. In other words, every $x \in U$ is connected to x_0 via a path in U, but then, U itself is path-connected, since given any two $x, y \in U$, we can combine the paths from x to x_0 and from x_0 to y to obtain a path from x to y.

Thus, open connected subsets of $\mathbb{R}^n$ are path connected. ■

Q5

Let X be an uncountable set.

(a) Show that the finite complement topology on X is compact.
(b) Is the countable-complement topology on X compact?

Answer.

(a) Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X. Fix $U_{\alpha_0} \in \mathcal{U}$. Since U_{α_0} is open, its complement is finite, so write $U_{\alpha_0}^c = \{x_1, ..., x_n\}$ and let, for each $1 \le i \le n$, α_i be such that $x_i \in U_{\alpha_i}$. Then,

$$X = U_{\alpha_0} \cup U_{\alpha_0}^c \subset U_{\alpha_0} \cup \left(U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}\right)$$

so that $\left\{U_{\alpha_0}, ..., U_{\alpha_n}\right\}$ is a finite subcover of $\mathcal{U}$.

Thus, every open cover of X has a finite subcover, so X is compact.

(b) **No.** We will exhibit a closed subspace of X which is not compact. First, we show that subspaces of X also have the countable-complement topology.

Lemma. *Let $S \subset X$ have the subspace topology. Then, the topology on S is the countable-complement topology.*

Proof. Let $U \subset S$ be open in the subspace topology. Then, $U = S \cap U'$ where U' is open in X, which is to say either $U' = \emptyset$ in which case $U = \emptyset$ is open, or $X \setminus U'$ is countable so that

$$S \setminus U = S \setminus U' \subset X \setminus U'$$

must also be countable, so U is open in the countable-complement topology on S.

Conversely, suppose $U \subset S$ is open in the countable-complement topology. If $U = \emptyset$, then obviously U is open in the subspace topology on S as well. Otherwise, $S \setminus U$ is countable and $U = S \cap (S^c \cup U)$ where $S^c \cup U$ is open in X because

$$X \setminus (S^c \cup U) = (X \setminus S^c) \cap X \setminus U = S \setminus U$$

is countable.

Thus, the subspace topology on S is equal to the countable complement topology on S. ■

Since X is infinite, it has a countable subspace, call it $\mathcal{N} = \{x_1, x_2, x_3, ...\}$.

Note that $\mathcal{N}$ is closed, since its complement has a countable complement (namely, $\mathcal{N}$) and hence is open.

Now, for each x_i, the singleton $\{x_i\}$ has countable $\mathcal{N}$-complement, so is open in the subspace $\mathcal{N}$, by our lemma. Therefore, the set $\mathcal{U} = \{\{x_i\}\}_{i=1}^{\infty}$ forms an open cover of $\mathcal{N}$. But, $\mathcal{U}$ can have no finite subcover, since any finite union of elements in $\mathcal{U}$ is finite, and therefore can not contain $\mathcal{N}$.

So, $\mathcal{N}$ is closed but not compact. Since X has a closed but not compact subspace, we have by the contrapositive of the "closed subspaces of compact spaces are compact" theorem from class that X is not compact. ■

Q6

Show that every compact subspace A of a metric space M is closed and bounded.

Is the converse true?

Answer.

Let M be a metric space and $A \subset M$ be compact.

We show boundedness first. Fix some $x_0 \in M$ and consider $\mathcal{U} = \{B_r(x_0) : r > 0\}$. Clearly, $\mathcal{U}$ is an open cover of A, since given any $a \in A$, the distance $d(a, x_0)$ is finite and hence in $B_r(x_0)$ for some $r > 0$. By compactness of A, $\mathcal{U}$ has a finite subcover $B_{r_1}(x_0), ..., B_{r_n}(x_0)$. Taking $r = \max\{r_1, ..., r_n\}$, we then have $A \subset B_r(x_0)$, so A is bounded.

For closedness of A, recall that every metric space is Hausdorff, since given any two points x, y, the balls of radius $\frac{d(x,y)}{2}$ centered at x, y separate them, and we showed in class that compact subspaces of Hausdorff spaces are closed.

Thus, every compact subspace A of a metric space M is closed and bounded. ■

No, the converse is not true. Let $M = \mathbb{R}$ with the bounded metric, i.e

$$d(x, y) = \min(|x - y|, 1)$$

Under this metric, $\mathbb{R}$ is a closed and bounded subset of itself, but is not compact because the above metric induces the standard topology on $\mathbb{R}$ under which $\mathbb{R}$ is not compact, as was shown in class.

Q7

Show that if A and B are disjoint compact subsets of a Hausdorff space X, then there exists disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Answer.

We first show that we can separate individual points of A from the entirety of B.

Claim. *For each $x \in A$, there exist disjoint open sets U_x, V_x, such that $x \in U_x$ and $B \subset V_x$.*

Proof. Let $x \in A$. For each $y \in B$, we have $x \neq y$, since A, B are disjoint, so choose by the Hausdorff condition some disjoint open sets Φ_y, Ψ_y such that $x \in \Phi_y, y \in \Psi_y$.

Then, the collection of all Ψ_y form an open cover of B, so by compactness of B, we get $y_1, ..., y_n$ so that $\Psi_{y_1}, ..., \Psi_{y_n}$ cover B. Set

$$U_x = \bigcap_{i=1}^{n} \Phi_{y_i} \quad \text{and} \quad V_x = \bigcup_{i=1}^{n} \Psi_{y_i}$$

Clearly, U_x, V_x are open, $x \in U_x$, and $B \subset V_x$. Also, U_x, V_x are disjoint, for if $b \in V_x$, then $b \in \Psi_{y_i}$ for some i and so $b \notin \Phi_{y_i}$ by construction, and hence $b \notin U_x$. ■

For each $x \in A$, let U_x, V_x be as in the above claim. Then, the collection of all U_x form an open cover of A, so by compactness of A, we get $x_1, ..., x_n$ such that $U_{x_1}, ..., U_{x_n}$ cover A. Then, set

$$U = \bigcup_{i=1}^{n} U_{x_i} \quad \text{and} \quad V = \bigcap_{i=1}^{n} V_{x_i}$$

Clearly, U, V are open. By choice of the x_i, we have $A \subset U$, and since $B \subset V_{x_i}$ for each i, we also have $B \subset V$. Moreover, U, V are disjoint, for if $a \in U$, then $a \in U_{x_i}$ for some i and so $a \notin V_{x_i}$ by construction, and hence $a \notin V$. ■