MAT327 – HW4

2024-10-09

Question 2

Show that a topological space X is Hausdorff iff the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.

Answer.

($\Longrightarrow$) Let X be a Hausdorff space and suppose for the sake of contradiction that Δ is not closed, so that there exists $(x,y) \in \overline{\Delta}$ but $(x,y) \notin \Delta$. Since $(x,y) \notin \Delta$, $x \neq y$, and thus by the Hausdorff condition we have open neighborhoods $U, V \subset X$ of x, y respectively such that $U \cap V = \emptyset$. But then, $U \times V$ is an open neighborhood of (x,y) in $X \times X$ so, since $(x,y) \in \overline{\Delta}$, $U \times V$ intersects Δ, meaning we have some $(a,a) \in U \times V$. Of course, this means a in both U and V, contradicting disjointness of U, V. Thus, Δ is closed in $X \times X$.

($\Longleftarrow$) Suppose the diagonal is closed in $X \times X$. Let $x, y \in X$ and suppose $x \neq y$. Then, $(x,y) \notin \Delta$ and, since $\Delta = \overline{\Delta}$, $x, y \notin \overline{\Delta}$ so there exists some basic open neighborhood $U \times V$ of (x,y) such that $U \times V$ does not intersect Δ. That is, U is an open neighborhood of x and V is an open neighborhood of y but $U \cap V = \emptyset$. Since every distinct pair of points in X have disjoint open neighborhoods, X is Hausdorff.

Both implications show the desired result: X is Hausdorff iff Δ is closed in $X \times X$. $\blacksquare$

Question 3

If U is an open set, is it true that $U = \text{Int}\,\overline{U}$?

Answer. No, this is not true, though it is true that $U \subset \text{Int}\,\overline{U}$, since U is an open set contained in $\overline{U}$ and $\text{Int}\,\overline{U}$ is the union of all such open sets.

To see that equality need not hold, consider the set $U = (0, 1) \cup (1, 2)$ as a subset of $\mathbb{R}$ in the standard topology. Then, $\overline{U} = [0, 2]$ so $\text{Int}\,\overline{U} = (0, 2)$ which is not equal to U.

Question 4

Let Y be an ordered set taken with the order topology, and assume $f, g : X \to Y$ are continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.
(b) Let $h(x) := \max(f(x), g(x))$. Show that h is a continuous function.

Answer. Let X, Y be topological spaces with Y having the order topology.

(a) Let $f, g : X \to Y$ be continuous functions and set

$$S = \{x : f(x) \leq g(x)\}$$

Assume for the sake of contradiction that S is not closed so that there exists $x_0 \in X$ such that $x_0 \in \overline{S}$ but $x_0 \notin S$; which is to say $f(x) > g(x)$.

We take cases on whether the interval $(g(x), f(x))$ is empty in Y, though in either case, the spirit of the argument remains the same: we construct two neighborhoods G, F of x_0 such that everything in the g-image of G is less than everything in the f-image of F and obtain a contradiction by considering $G \cap F$. See Figure 1.

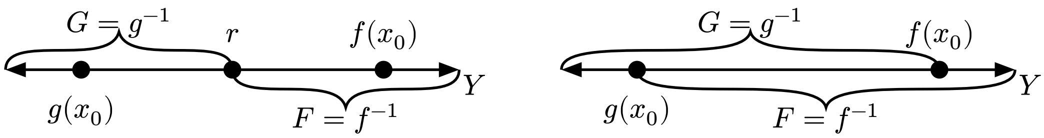

Figure 1: Construction of G, F if $(g(x_0), f(x_0)) \neq \emptyset$) (left) or otherwise (right)

Suppose $(g(x_0), f(x_0))$ is not empty, so that there exists $r \in Y$ with $g(x_0) < r < f(x_0)$. Set $G = g^{-1}((-\infty, r))$ and $F = f^{-1}((r, \infty))$. Notice that $x_0 \in G$ since $g(x_0) < r$ and similarly $x_0 \in F$ since $f(x_0) > r$. Moreover, G, F are open since they're continuous preimages of open sets. Thus, $G \cap F$ is an open neighborhood of x_0 and hence, since $x_0 \in \overline{S}$, there exists $x \in G \cap F$ such that $x \in S$. Since $x \in S$, we must have $f(x) \leq g(x)$, but since $x \in G \cap F$, we also have $g(x) < r < f(x)$. This is a contradiction.

Suppose $(g(x_0), f(x_0))$ is empty. Then, for every $y \in Y$, either $y \leq g(x_0)$ or $y \geq f(x_0)$. Set $G = g^{-1}((-\infty, f(x_0)))$ and $F = f^{-1}((g(x_0), \infty))$. Again, notice that both G, F are open, being continuous preimages of open sets and both contain x_0, since $g(x_0) < f(x_0)$. Thus, $G \cap F$ is an open neighborhood of x_0 and so, since $x_0 \in \overline{S}$, there exists $x \in G \cap F$ such that $x \in S$, meaning $f(x) \leq g(x)$. On the other hand, $x \in G$ so $g(x) < f(x_0)$ and hence, since $(g(x_0), f(x_0))$ is empty, $g(x) \leq g(x_0)$. But, $x \in F$ implies $f(x) > g(x_0)$ so that

$$g(x) \leq g(x_0) < f(x)$$

which again is a contradiction.

In either case, we obtain a contradiction, so we conclude that S is closed. ∎

(b) Let $f, g : X \to Y$ be continuous. Set

$$A = \{x : f(x) \leq g(x)\}$$
$$B = \{x : g(x) \leq f(x)\}$$

Define

$$h_A : A \to Y \qquad h_A(x) = g(x)$$
$$h_B : B \to Y \qquad h_B(x) = f(x)$$

Now, by (a), both A, B are closed. Also, since $h_A = g|_A$ and $h_B = f|_B$, and restrictions of continuous functions are continuous, both h_A, h_B are continuous. Moreover, if $x \in A \cap B$, then both $f(x) \leq g(x)$ and $g(x) \leq f(x)$ which, by anti-symmetry, implies

$$h_A(x) = g(x) = f(x) = h_B(x)$$

Finally, observe that $A \cup B = X$, since linearity of the order on Y implies that for every x, either $f(x) \leq g(x)$ or $g(x) \leq f(x)$.

Thus, since A, B are closed sets partitioning X and h_A, h_B are continous functions out of A, B agreeing on their intersection, the Pasting Lemma implies that the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} h_A(x) & x \in A \\ h_B(x) & x \in B \end{cases}$$

is a well-defined continuous function.

We'll show that in fact, $h(x) = \max(f(x), g(x))$. Indeed, for every $x \in X$, if $f(x) \leq g(x)$, then $x \in A$ so $h(x) = h_A(x) = g(x)$ and $\max(f(x), g(x)) = g(x)$. Otherwise, $g(x) < f(x)$ so that $x \in B$ and hence $h(x) = h_B(x) = f(x)$ and $\max(f(x), g(x)) = f(x)$. In either case, $h(x)$ and $\max(f(x), g(x))$ agree.

Thus, if $f, g : X \to Y$ are continuous, then the function $h(x) = \max(f(x), g(x))$ is continuous. ■

Question 5

Show that if X is a Hausdorff space and if $x_1, ..., x_n$ are distinct points of X, then there exist open sets $U_1, ..., U_n$ in X such that for every i, $x_i \in U_i$ and such that if $i \neq j$, then $U_i \cap U_j = \emptyset$.

Answer. We proceed by induction on $n \in \mathbb{Z}^+$.

In the base case, $n = 1$, we take $U_1 = X$ so that $x_1 \in U$ and there are no $i \neq j$.

Now, let $n \in \mathbb{Z}^+$ be arbitrary and suppose that for every distinct $x_1, ..., x_n \in X$, there exist pairwise disjoint open neighborhoods of every x_i.

Let $x_1, ..., x_{n+1}$ be distinct points of X. Applying the induction hypothesis to $x_1, ..., x_n$, we obtain pairwise disjoint open sets $U_1', ..., U_n'$ such that $x_i \in U_i'$ for $1 \leq i \leq n$. By the Hausdorff condition, we have for each $1 \leq i \leq n$, some disjoint open sets V_i, W_i such that $x_i \in V_i$ and $x_{n+1} \in W_i$. Then, we set

$$U_i = U_i' \cap V_i \text{ for } 1 \leq i \leq n \qquad \text{and} \qquad U_{n+1} = \bigcap_{i=1}^{n} W_i$$

Since finite intersections of open sets are open, each U_i is open. Moreover, for every $1 \leq i \leq n$, we have $x_{n+1} \in W_i$ and x_i in both U_i', V_i so U_i is an open neighborhood of x_i and U_{n+1} is an open neighborhood of x_{n+1}.

It remains to show pairwise disjointness. Let $1 \leq i < j \leq n + 1$.

If $j < n + 1$, then we have

$$\begin{aligned} U_i \cap U_j &= (U_i' \cap V_i) \cap (U_j' \cap V_j) \\ &= (U_i' \cap U_j') \cap (V_i \cap V_j) \\ &= \emptyset \cap (V_i \cap V_j) && U_i', U_j' \text{ are disjoint} \\ &= \emptyset \end{aligned}$$

If $j = n + 1$, then

$$\begin{aligned} U_i \cap U_j &= (U_i' \cap V_i) \cap \bigcap_{k=1}^{n} W_k \\ &= (V_i \cap W_i) \cap U_i' \cap \bigcap_{\substack{k=1 \\ k\neq i}} W_k \\ &= \emptyset \cap U_i' \cap \bigcap_{\substack{k=1 \\ k\neq i}} W_k && V_i, W_i \text{ are disjoint} \\ &= \emptyset \end{aligned}$$

In either case, we get U_i, U_j are disjoint.

By induction, if X is a Hausdorff space, then for every distinct $x_1, ..., x_n \in X$, there exist pairwise disjoint open neighborhoods of each x_i. ■

Disclaimers (added October 9th)

- In 4a), I argued that S is closed by contradiction. An alternative approach, which I think is a little cleaner, is to argue directly that $X \setminus S$ is open. We can write

$$\begin{aligned} X \setminus S &= \{x : f(x) > g(x)\} \\ &= \bigcup_{y \in Y} \left(g^{-1}((-\infty, y)) \cap f^{-1}((y, \infty))\right) \\ &\cup \bigcup_{\substack{y < y' \in Y \\ (y,y') = \emptyset}} \left(g^{-1}((-\infty, y')) \cap f^{-1}((y, \infty))\right) \end{aligned}$$

  The $(\subset)$ inclusion is straightforward by taking cases on the existence of some $y \in (f(x), g(x))$.

  The $(\supset)$ is also pretty short. If x is in the first union (indexed over y), then we immediately get $x \in X \setminus S$ by transitivity of the order. Otherwise, we argue much as in 4a) that $g(x) \leq y < f(x)$.

  In either approach, we have to do that thing about taking cases on whether two elements of Y have a third between them. I find that annoying but I don't know if there's a way around it :(

- In Q5, I argued by induction which, as Kai said in his feedback, really doesn't give us much. Directly choosing open sets for every pair of distinct points and intersecting them is much simpler. One way of setting that up is to choose for each $1 \leq i, j \leq n$ such that $i \neq j$, some open sets V_{ij}, W_{ij} such that $x_i \in V_{ij}, x_j \in W_{ij}$, and $V_{ij} \cap W_{ij} = \emptyset$. We get our open neighborhood of x_k by setting

$$U_k = \bigcap_{\substack{\ell=1 \\ k \neq \ell}}^{n} V_{k\ell} \cap \bigcap_{\substack{\ell=1 \\ k \neq \ell}}^{n} W_{\ell k}$$

  Then, $x_k \in U_k$ since x_k is in each V_{kj} and each W_{ik} and if $i \neq j$, then

$$\begin{aligned} U_i \cap U_j &= \left(\bigcap_{\substack{\ell=1 \\ i \neq \ell}}^{n} V_{i\ell} \cap \bigcap_{\substack{\ell=1 \\ i \neq \ell}}^{n} W_{\ell i} \right) \cap \left(\bigcap_{\substack{\ell=1 \\ j \neq \ell}}^{n} V_{j\ell} \cap \bigcap_{\substack{\ell=1 \\ j \neq \ell}}^{n} W_{\ell j} \right) \\ &= V_{ij} \cap W_{ij} \cap \ldots \\ &= \emptyset \end{aligned}$$

  so the U_k are pairwise disjoint.