MAT327H1 F Homework Assigment 3

October 3, 2024

Question 1. *Do readings.*

Question 2.

Proof. We show that both π_X and π_Y are open maps. Let U be open in $X \times Y$ under the product topology. So U is a union of some indexed family of basic sets $\{B_\alpha\}_\alpha$, $U = \bigcup_{B \in \mathcal{B}} B$. By definition of the product topology, each basic set B_α is of the form $V_\alpha \times W_\alpha$ where both V_α and W_α are open in X and Y respectively. But then we have the following (using what was shown in HW1 about the unions of images):

$$\pi_X(U) = \pi_X(\bigcup U_\alpha) = \bigcup \pi_X(U_\alpha) = \bigcup \pi_X(W_\alpha \times V_\alpha) = \bigcup W_\alpha \tag{1}$$

$$\pi_Y(U) = \pi_Y(\bigcup U_\alpha) = \bigcup \pi_Y(U_\alpha) = \bigcup \pi_Y(W_\alpha \times V_\alpha) = \bigcup V_\alpha \tag{2}$$

And we see that both of these sets are unions of basic sets, which are both open. □

Question 3.

First let us consider the special case where L is a straight vertical line. For fixed $x_0 \in \mathbb{R}$, L is of the form $L = \{x_0 \times y \mid y \in \mathbb{R}\}$. Let us use Lemma 16.1 to describe the basis that L inherits from $\mathbb{R}_\ell \times \mathbb{R}$, and $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

As a subspace of $\mathbb{R}_\ell \times \mathbb{R}$, a basis for the subspace topology on L is given by the collection $\mathcal{B}_L = \{B \cap L \mid B \in \mathcal{B}\}$. Theorem 15.1 tells us that the sets B are of the form $[a, b) \times (c, d)$ where $a < b, c < d \in \mathbb{R}$. It follows that the basis $\mathcal{B}_L$ is a collection of sets of the form $\{x_0\} \times (c, d)$. As a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$, a basis for the subspace topology on L is given by the collection $\mathcal{B}'_L = \{B' \cap L \mid B' \in \mathcal{B}'\}$. Theorem 15.1 tells us that the sets B' are of the form $[a, b) \times [c, d)$ where $a < b, c < d \in \mathbb{R}$. It follows that the basis $\mathcal{B}'_L$ is a collection of sets of the form $\{x_0\} \times [c, d)$. The function $\varphi(x \times y) = y$ maps $\{x_0\} \times (c, d) \to (c, d)$ and $\{x_0\} \times [c, d) \to [c, d)$ and defines a homeomorphism between the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}$, and the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and $\mathbb{R}_\ell$.

So, as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$, L inherits what is essentially the standard topology on $\mathbb{R}$. As a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$, L inherits what is essentially the lower limit topology, $\mathbb{R}_\ell$

In the more general case let us define $L = \{(x \times mx + b \mid x \in \mathbb{R}\}$ for some fixed $m, b \in \mathbb{R}$. We consider first how L behaves as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$. By Lemma 16.1, the basis for the topology are the sets of the form $[(a, ma + b), (c, mc + b))$, $((a, ma + b), (c, mc + b))$ for $a, c \in \mathbb{R}$ and $a < c$. Now we can define a homeomorphism; $\varphi : L \cap (\mathbb{R}_\ell \times \mathbb{R}) \to \mathbb{R}_\ell$ by $\varphi(a, ma + b) = a$, which gives the following mappings between basic sets;

$$((a, ma + b), (c, mc + b)) \to (a, c),$$

$$[(a, ma + b), (c, mc + b)) \to [a, c).$$

We claim this defines a homeomorphism with $\mathbb{R}_\ell$. It is easy to see that it is a bijection as there exists an inverse simply by mapping $a \to (a, ma + b)$. It is also continuous, as basic sets of $\mathbb{R}_\ell$ have preimages that are basic sets in the topology on L. Likewise, it has a continuous inverse because basic sets of L map to sets that are open in $\mathbb{R}_\ell$. Both of these kinds of sets are open because the topology of $\mathbb{R}_\ell$ is strictly finer than that of $\mathbb{R}$ (Lemma 13.4).

For $\mathbb{R}_\ell \times \mathbb{R}_\ell$, we can apply a similar logic, but we have two cases. If L has a finite slope, then we must consider $m \geq 0, m < 0$. The case $m \geq 0$ is similar to the one above, but we only have to consider basic sets of the form $[a, b)$. So as a subspace of $\cap(\mathbb{R}_\ell \times \mathbb{R}_\ell)$, L is homeomorphic to $\mathbb{R}_\ell$. When $m < 0$, notice that $\forall (x, y) \in L\ \exists [x, a) \times [y, b) \in (\mathbb{R}_\ell \times \mathbb{R}_\ell)$ such that $L \cap [x, a) \times [y, b) = \{(x, y)\}$. This means that open sets in this topology are of the form $\{(x, y)\}$ by Lemma 16.1. It is not difficult to see that the topology on L is homeomorphic to the discrete topology on $\mathbb{R}$. *(This question asked for an explanation, not a proof, so I have skipped over some of the details such as checking in each case that there indeed exists a homeomorphism. I hope this is acceptable. Also, please see the photo for reference)*

Question 4.

Proof. Define $\mathcal{B}$ and $\mathcal{B}'$ to be bases for the dictionary order topology (on $\mathbb{R} \times \mathbb{R}$) and the product topology (on $\mathbb{R}_d \times \mathbb{R}$) respectively. By definition, basic sets $B \in \mathcal{B}$ are of the form $(a \times b, c \times d)$ for $a \leq c$, and if $a = c, b < d$. Theorem 15.1 tells us that basic sets $B' \in \mathcal{B}'$ are of the form $\{x_1\} \times (y_1, y_2), y_1 < y_2$, where $\{x_1\}$ and (y_1, y_2) are basis elements in the discrete and standard topology respectively. It is then easy to show that these topologies are both finer than each other by applying Lemma 13.3.

First we show that $\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$. Given any point $x_0 \times y_0 \in (a \times b, c \times d)$, the set $(x_0 \times y_0 - |y_0 - b|/2, x_0 \times y_0 + |d - y_0|/2)$ contains $x_0 \times y_0$ and is contained in $(a \times b, c \times d)$. But it is clear that this is just a set of the form $\{x_0\} \times (y_0 - |y_0 - b|/2, y_0 + |d - y_0|/2)$. Which is clearly open in the product topology $\mathbb{R}_d \times \mathbb{R}$. Lemma 13.3 tells us that the product topology is finer than the dictionary order topology. Conversely, given any basic set $\{x_0\} \times (a, b)$ in the product topology and a point $x \in \{x_0\} \times (a, b)$, the set $(x_0 \times a, x_0 \times b)$ is open in the dictionary order topology, contains x and is contained in $\{x_0\} \times (a, b)$ (the two are actually equal). Lemma 13.3 implies that the dictionary order topology is finer than the product topology.

We conclude that the two are equal.

I claim that this topology is strictly finer than the standard topology on $\mathbb{R}^2$. To show that it is finer, notice that of course any basic set of $\mathbb{R}^2$ (under the standard topology call it $\mathcal{T}$) is contained in "our special topology". If $(a, b) \times (c, d) \in \mathbb{R}^2$ is a basic set, then $\bigcup_{x \in (a,b)} \{x\} \times (c, d)$ is an open set in our special topology. On the other hand, $\{x\} \times (c, d) \notin \mathcal{T}$ as this would imply that $\{x\}$ is an open set in $\mathbb{R}$ under the standard topology, which we know is not true. $\square$

Question 5.

We can view the identity function in one direction as $\mathrm{id} : A \times B_{\text{sub of prod}} \to A \times B_{\text{prod of sub}}$ given by $\mathrm{id} = (\pi_X \circ i_A, \pi_Y \circ i_B)$, which is a composition of continuous maps, thrown together into a tuple. This is continuous by a few Theorems in the book. In the other direction, we can view it as $\mathrm{id} = (\pi_X \circ i_A \circ \pi_A, \pi_Y \circ i_B \circ \pi_B)$. A proof of this (probably wrong) is shown below: