# MAT327H1 F Homework Assigment 2

October 3, 2024

**Question 1.** *Do readings.*

**Question 2.**

*Proof.* Let $X$ be a topological space, and $A \subset X$. Assume that $\forall x \in A$, $\exists U \subset X$ open s.t. $x \in U \subset A$. We show that $A$ is open in $X$. For each $x \in A$, denote $U_x$ as the open set s.t. $x \in U_x$ and $U_x \subset A$. Notice that $\forall x \in A$, $\{x\} \subset U_x$, so it must be that $A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x$. Conversely, we have assumed that each $U_x \subset A$, so it must follow that $\bigcup_{x \in A} U_x \subset A$. This implies that $A = \bigcup_{x \in A} U_x$, which is the union of sets which are open in $X$. Therefore, $A$ is open in $X$. □

**Question 3.**

(a)

*Proof.* We show that $\mathcal{T}_c$ is indeed a topology on $X$. If $X$ is countable, then it is clear that $\mathcal{T}_c$ is the discrete topology on $X$, as every set must have a countable complement. Suppose that $X$ is nonempty and uncountable. $\varnothing \in \mathcal{T}_c$ by definition, and $X \in \mathcal{T}_c$ because $X - X = \varnothing$ is countable. Now suppose that $U_\alpha$ is an indexed family of open sets of $X$, with index set $J$. So $\forall \alpha,\ X - U_\alpha$ is countable. We show that $\bigcup_{\alpha \in J} U_\alpha$ is open in $X$. By DeMorgan's Law;

$$X - \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (X - U_\alpha)$$

and so of course, for any $\alpha_0 \in J$, $\bigcap_{\alpha \in J}(X - U_\alpha) \subset X - U_{\alpha_0}$ meaning that $\bigcap_{\alpha \in J}(X - U_\alpha)$ is a subset of a countable set, and is thus countable. Therefore $\bigcup_{\alpha \in J} U_\alpha$ is open in $X$. Now suppose that $U_i, 1 \leq i \leq n$ is a finite collection of open subsets of $X$. We show that $\bigcap_{i=1}^n U_i$ is open in $X$. Once again, by DeMorgan's Law;

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

where each $X - U_i$ is countable. A finite union of countable sets is countable, so it follows that $\bigcap_{i=1}^n U_i$ is indeed open in $X$. We have shown that the three sufficient conditions hold for $\mathcal{T}_c$ to be a topology on $X$, so we are done. $\square$

(b) We claim that $\mathcal{T}_\infty$ is not a topology on $X$.

*Proof.* Let $X = \mathbb{R}$ be equipped with the infinite-complement "topology", and fix $x \in \mathbb{R}$. Consider the sets $U_1, U_2 \subset \mathbb{R}$ given by $U_1 = (-\infty, x),\ U_2 = (x, \infty)$. $U_1^c = [x, \infty)$ and $U_2^c = (-\infty, x]$ are both infinite, so indeed $U_1, U_2 \in \mathcal{T}_\infty$. However, $U_1 \cup U_2$ is not open in $\mathbb{R}$ under this topology, as the complement is finite; $(U_1 \cup U_2)^c = ((-\infty, x) \cup (x, \infty))^c = (\mathbb{R} - \{x\})^c = \{x\}$. Therefore $\mathcal{T}_\infty$ fails (in this case) to satisfy that arbitrary unions of open sets are open, and is thus not a topology. $\square$

**Question 4.**

(a)

*Proof.* Suppose that $\{\mathcal{T}_\alpha\}$ is a family of topologies on $X$. We show that $\bigcap_\alpha \mathcal{T}_\alpha$ is also a topology on $X$.

By definition of a topology we have that $\forall \mathcal{T} \in \{\mathcal{T}_\alpha\}$, $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$. Therefore $X \in \bigcap_\alpha \mathcal{T}_\alpha$ and $\varnothing \in \bigcap_\alpha \mathcal{T}_\alpha$.

Next, let $U_\beta$ be a collection of subsets of $X$ s.t. $\forall \beta$, $U_\beta \in \bigcap_\alpha \mathcal{T}_\alpha$. We show that $\bigcup_\beta U_\beta \in \bigcap_\alpha \mathcal{T}_\alpha$. Firstly, since $\forall \beta$, $U_\beta \in \bigcap_\alpha \mathcal{T}_\alpha$, it follows that $\forall \beta$, $U_\beta \in \mathcal{T}_\alpha$ for every $\alpha$. Since $U_\beta \in \mathcal{T}_\alpha$ for every $\alpha$, $\bigcup_\beta U_\beta \in \mathcal{T}_\alpha$ for every $\alpha$, which follows from the fact that each $\mathcal{T}_\alpha$ is a topology. Therefore, $\bigcup_\beta U_\beta \in \bigcap_\alpha \mathcal{T}_\alpha$.

Finally, let $U_i$, $1 \leq i \leq n$ be a finite collection of subsets of $X$ s.t. $U_i \in \bigcap_\alpha \mathcal{T}_\alpha$. Then each $U_i \in \mathcal{T}_\alpha$ for every $\alpha$. Since each $\mathcal{T}_\alpha$ is a topology, we know that $\bigcap_i U_i \in \mathcal{T}_\alpha$ for each $\alpha$. Therefore, $\bigcap_i U_i \in \bigcap_\alpha \mathcal{T}_\alpha$ as needed.

This shows that indeed $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on $X$. $\square$

(b)

*Proof.* We claim that $\bigcap_\alpha \mathcal{T}_\alpha$ is the unique largest topology contained in all of the $\mathcal{T}_\alpha$'s. We have already verified in part (a) of the question that this is indeed a topology on $X$. First we show that it is the largest, meaning that if $\mathcal{T}'$ is another topology on $X$ which is contained in all the $\mathcal{T}_\alpha$'s, we show that $\bigcap_\alpha \mathcal{T}_\alpha \supset \mathcal{T}'$. Let $U \in \mathcal{T}'$. Since $\mathcal{T}'$ is contained in all the $\mathcal{T}_\alpha$'s, it must be that $\mathcal{T}'$ is also contained in their intersection, $\bigcap_\alpha \mathcal{T}_\alpha$. To show uniqueness, suppose there are two largest topologies contained in all the $\mathcal{T}_\alpha$'s; $\mathcal{T}_1$ and $\mathcal{T}_2$. It then follows that $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_2 \subset \mathcal{T}_1$, which implies that $\mathcal{T}_2 = \mathcal{T}_1$ - not unique. This shows that indeed $\bigcap_\alpha \mathcal{T}_\alpha$ is the unique largest topology contained in all of the $\mathcal{T}_\alpha$'s.

Now we show that there exists a smallest topology containing all of the $\mathcal{T}_\alpha$'s. Define the collection $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$, which is a collection of subsets of $X$ whose union is $X$. Using $\mathcal{S}$, define the topology $\mathcal{T}_s$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$. In the textbook, this is called *the topology generated by the subbasis* $\mathcal{S}$. Let us check that this is indeed a topology on $X$. It is enough to check that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis, from which it will follow by a theorem proven in class (Lemma 13.1 in the textbook) that the collection of all unions of elements of $\mathcal{B}$ is a topology - which is of course the topology $\mathcal{T}_s$.

Since the union of elements of $\mathcal{S}$ is $X$, we must have that for each $x \in X$, there is at least one $B \in \mathcal{B}$ containing $x$. Now suppose that $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$. Since both $B_1$ and $B_2$ are finite intersections of elements of $\mathcal{S}$, so is the intersection $B_1 \cap B_2$. This means that the set $B_3 = B_1 \cap B_2$ is itself a basic set which is of course contained in itself, showing that $\mathcal{B}$ is indeed a basis - and thus $\mathcal{T}_s$ is indeed a topology.

Now we show that $\mathcal{T}_s$ is the smallest topology containing all the $\{\mathcal{T}_\alpha\}$'s. Let $\mathcal{T}''$ be another topology containing all the $\{\mathcal{T}_\alpha\}$'s, and let $U \in \mathcal{T}_s$. We show that $U \in \mathcal{T}''$. Since $\mathcal{T}''$ contains all of the $\{\mathcal{T}_\alpha\}$'s, it must also contain all of the unions of finite intersections of elements (open sets) in the collection $\bigcup_\alpha \mathcal{T}_\alpha$. This follows from the assumption that $\mathcal{T}''$ is a topology. Since $U$ is open in $\mathcal{T}_s$, it is a union of basic sets, and the basic sets are finite intersections of elements of $\bigcup_\alpha \mathcal{T}_\alpha$, meaning that $U$ is a union of finite intersections of elements of $\bigcup_\alpha \mathcal{T}_\alpha$ (this is also obvious from the definition of *topology generated by the subbasis* $\mathcal{S}$. So $U \in \mathcal{T}''$.

To show uniqueness, suppose that there exists a different smallest topology containing all of the $\{\mathcal{T}_\alpha\}$'s, call it $\mathcal{T}'''$. So by this assumption, $\mathcal{T}_s \supset \mathcal{T}'''$, but from what we have already shown, $\mathcal{T}_s \subset \mathcal{T}'''$ so it follows that $\mathcal{T}_s$ is unique. □

(c) We use what we have shown in the previous part. The largest topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T}_1 \cap \mathcal{T}_2 = \{\varnothing, X, \{a\}\}$. Following the same notation as in part (b), for the smallest topology containing both $\mathcal{T}_1$ and $\mathcal{T}_2$, we have $\mathcal{S} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}\}$, so $\mathcal{B} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. And the set of all unions of elements in $\mathcal{B}$ is then $\mathcal{T}_s = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.

**Question 5.**

(a)

*Proof.* Denote $\mathcal{B}_{\mathbb{Q}} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$. To show that $\mathcal{B}_{\mathbb{Q}}$ is a basis for the standard topology on $\mathbb{R}$, it suffices to show that $\mathcal{B}_{\mathbb{Q}}$ satisfies the assumptions on $\mathcal{C}$ in the following lemma:

*Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of $X$ such that for each open set $U$ of $X$ and each $x$ in $U$, there is an element $C$ of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of $X$.*

This lemma was proven in class, and is stated as Lemma 13.2 in the textbook. So let $x \in \mathbb{R}$, and let $U$ be an open set containing $x$, under the standard topology on $\mathbb{R}$. Let $(a_0, b_0) \subset \mathbb{R}$ where $a_0 < b_0 \in \mathbb{R}$ be a basis element (in the standard topology on $\mathbb{R}$) containing $x$ which is contained in $U$. We know such a set $(a_0, b_0)$ exists because the standard topology on $\mathbb{R}$ is generated by basic sets of this form.

So, it follows that $x > a_0 \wedge x < b_0$, so there exists $p_1, q_1 \in \mathbb{Q}$ s.t. $p_1 \in (a_0, x) \wedge q_1 \in (x, b_0)$ (because the rationals are dense in the reals). So the basis element $B = (p_1, q_1) \in \mathcal{B}_{\mathbb{Q}}$ contains $x$, and is contained in $U$, as needed.

So, the assumptions of Lemma 13.2 are satisfied, which shows that $\mathcal{B}_{\mathbb{Q}}$ is a basis for the standard topology on $\mathbb{R}$. □

(b)

*Proof.* First we show that the collection is indeed a basis for some topology on $\mathbb{R}$. For any $x \in \mathbb{R}$ there surely exists $a, b \in \mathbb{Q}$ s.t. $x \in [a, b)$. Just take any $a < x < b$. Next, notice that the (non-empty) intersection of two basic sets is another basic set. Given $c < d, e < f \in \mathbb{Q}$, consider $[c, d) \cap [e, f)$. The intersection is simply $[\max\{c, e\}, \min\{d, f\})$, which is another basic set. So if $x$ is contained in the intersection of two basic sets, it is surely contained in a basic set which is a subset of the previous intersection, as the intersection is a basic set! (and of course fits inside itself). So the collection satisfies the definition of being a basis for some topology on $\mathbb{R}$.

Now we show that the topology it generates (call it $\mathcal{T}$) is different from the lower limit topology, which we will denote by $\mathcal{T}_\ell$. Consider the open set $[\pi, 4) \in \mathcal{T}_\ell$. We show that $[\pi, 4) \notin \mathcal{T}$. To do this, we can show that $\exists x \in [\pi, 4)$ where there is no basic set (in the basis which generates $\mathcal{T}$) that contains $x$ and is contained in $[\pi, 4)$. So suppose that $[p, q)$ is some basic set containing $x = \pi/4$, where $p < q \in \mathbb{Q}$. So we cannot have $p = x$, and also we cannot have $p < x$ because then $[p, q) \not\subset [\pi, 4)$. So $p > x$. But then $x \notin [p, q)$, so we conclude that such a basic set does not exist. Therefore, $[\pi, 4) \notin \mathcal{T}$, implying that the topology generated by this basis is indeed different from the lower limit topology on $\mathbb{R}$. □

**Question 6.**

*Proof.* ($\Longleftarrow$) We show that if $f$ is constant or finite-to-one, then $f$ must be continuous.

Let $U \subset Y$ open, meaning that $Y - U$ is either finite or all of $Y$. If it is all of $Y$, then there is not much to show in either case as this would imply $U = \varnothing$ and $f^{-1}(\varnothing) = \varnothing$, which is open in $X$. Let $U$ be non-empty, open in $Y$, then $Y - U$ must be finite, and we may denote it as $Y - U = \{y_1, \ldots, y_n\}$ for some $n \in \mathbb{N}$. If $f$ is constant, then for some $y_0 \in Y$, let $f(x) = y_0 \; \forall x \in X$ . Then if $y_0 \notin U$, then $f^{-1}(Y - U) = X$, which is open in $X$. If $y_0 \in U$ then $f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U) = Y - Y = \varnothing$, which is also open in $X$. In either case, $f$ is continuous. Now suppose that $f$ is finite-to-one. Then for each point $y_i \in Y - U$ there exist finitely many $x_{i1}, \ldots, x_{in} \in X$ s.t. $f(x_{ij}) = y_i \in Y - U$, $1 \leq j \leq n$, which implies the set $f^{-1}(Y - U)$ is finite. But recall from HW1 that $f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U) = X - f^{-1}(U)$, meaning $X - f^{-1}(U)$ is finite and thus $f^{-1}(U)$ is open in $X$. So $f$ is continuous.

($\Longrightarrow$) Now we show that if $f$ is continuous, $f$ must be constant or finite-to-one. Here we may assume that $X$ is infinite. If it were not, every function $g : X \to Y$ would be finite-to-one, so there would be nothing to show. We will proceed by showing the contrapositive statement; suppose that $f$ is neither constant nor finite-to-one and show that $f$ cannot be continuous. In other words, assume that $\exists y_0 \in Y$ s.t. $f^{-1}(\{y_0\})$ is infinite, and $\exists x_1, x_2 \in X$, $x_1 \neq x_2$ s.t. $f(x_1) \neq f(x_2)$. Now consider the open set $Y - \{y_0\} \subset Y$, which is open because the complement $\{y_0\}$ is finite. Notice that $f^{-1}(Y - \{y_0\}) = f^{-1}(Y) - f^{-1}(\{y_0\}) = X - f^{-1}(\{y_0\})$. Unless $X - f^{-1}(\{y_0\})$ is empty, it cannot be open as its complement, $f^{-1}(\{y_0\})$ is assumed to be infinite. If the set $X - f^{-1}(\{y_0\})$ were empty, then $f^{-1}(\{y_0\}) = X$, meaning that $f$ must be constant, which contradicts our assumption that $f$ is non-constant. Therefore the set $X - f^{-1}(\{y_0\})$ is not open, implying that $f$ cannot be continuous. This completes the proof. $\square$