1. (a) Suppose $a \in A_0$. Since

$$f^{-1}(f(A_0)) = \{x \in A : f(x) \in f(A_0)\}$$

and $a \in A$ satisfies $f(a) \in f(A_0)$, we have $a \in f^{-1}(f(A_0))$, and thus $A_0 \subseteq f^{-1}(f(A_0))$.

Suppose f is moreover injective and $a \in f^{-1}(f(A_0))$. Then $f(a) \in f(A_0)$, meaning there exists $x \in A_0$ such that $f(a) = f(x)$. By injectivity, $a = x$, so $a \in A_0$. Thus $f^{-1}(f(A_0)) \subseteq A_0$, so we have equality if f is injective.

(b) Suppose $b \in f(f^{-1}(B_0))$. Then $b = f(a)$ for some $a \in f^{-1}(B_0)$. Since

$$f^{-1}(B_0) = \{x \in A : f(x) \in B_0\},$$

$b = f(a) \in B_0$. Thus $f(f^{-1}(B_0)) \subseteq B_0$.

Suppose f is moreover surjective and $b \in B_0$. Then by surjectivity there exists $a \in A$ such that $f(a) = b$. Clearly $a \in f^{-1}(B_0)$, so $b = f(a) \in f(f^{-1}(B_0))$. Thus $B_0 \subseteq f(f^{-1}(B_0))$, so equality holds if f is surjective.

2. (a) Suppose $B_0 \subseteq B_1$ and $a \in f^{-1}(B_0)$. By definition, $f(a) \in B_0 \subseteq B_1$. Thus $a \in f^{-1}(B_1)$, showing that $f^{-1}(B_0) \subseteq f^{-1}(B_1)$.

(b) Suppose $a \in f^{-1}(B_0 \cup B_1)$. By definition $f(a) \in B_0 \cup B_1$; without loss of generality assume $f(a) \in B_0$. Then $a \in f^{-1}(B_0) \subseteq f^{-1}(B_0) \cup f^{-1}(B_1)$, showing that $f^{-1}(B_0 \cup B_1) \subseteq f^{-1}(B_0) \cup f^{-1}(B_1)$.

Conversely suppose $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$; without loss of generality assume $a \in f^{-1}(B_0)$. Then $f(a) \in B_0 \subseteq B_0 \cup B_1$, meaning $a \in f^{-1}(B_0 \cup B_1)$. Thus $f^{-1}(B_0) \cup f^{-1}(B_1) \subseteq f^{-1}(B_0 \cup B_1)$, so equality holds.

(c) $a \in f^{-1}(B_0 \cap B_1)$ is equivalent to $f(a) \in B_0 \cap B_1$. This occurs if and only if $f(a) \in B_0$ and $f(a) \in B_1$, or $a \in f^{-1}(B_0)$ and $a \in f^{-1}(B_1)$, respectively. Therefore $a \in f^{-1}(B_0 \cap B_1)$ if and only if $a \in f^{-1}(B_0) \cap f^{-1}(B_1)$, so $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$, as desired.

(d) $a \in f^{-1}(B_0 - B_1)$ is equivalent to $f(a) \in B_0 - B_1$, which we may write as $f(a) \in B_0$ and $f(a) \notin B_1$, or $a \in f^{-1}(B_0)$ and $a \notin f^{-1}(B_1)$. Therefore $a \in f^{-1}(B_0 - B_1)$ if and only if $a \in f^{-1}(B_0) - f^{-1}(B_1)$.

(e) Suppose $A_0 \subseteq A_1$ and $b \in f(A_0)$. Then $b = f(a)$ for some $a \in A_0$. By the inclusion, a is moreover in A_1, so $b = f(a)$ implies $b \in f(A_1)$. Therefore $f(A_0) \subseteq f(A_1)$, as desired.

(f) Suppose $b \in f(A_0 \cup A_1)$. Then $b = f(a)$ for some $a \in A_0 \cup A_1$; without loss of generality assume $a \in A_0$. Then $b = f(a)$ implies $b \in f(A_0) \subseteq f(A_0) \cup f(A_1)$, showing that $f(A_0 \cup A_1) \subseteq f(A_0) \cup f(A_1)$.

Conversely if $b \in f(A_0) \cup f(A_1)$, assume without loss of generality that $b \in f(A_0)$. Then $b = f(a)$ for some $a \in A_0$. Since $A_0 \subseteq A_0 \cup A_1$, we additionally have $a \in A_0 \cup A_1$, so $b = f(a)$ means that $b \in f(A_0 \cup A_1)$. Therefore $f(A_0) \cup f(A_1) \subseteq f(A_0 \cup A_1)$, showing that equality holds.

(g) Suppose $b \in f(A_0 \cap A_1)$. Then $b = f(a)$ for some $a \in A_0 \cap A_1$. In particular, $a \in A_0$ and $a \in A_1$, so $b = f(a)$ shows that $b \in f(A_0)$ and $b \in f(A_1)$, respectively. Thus $b \in f(A_0) \cap f(A_1)$, so that $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$.

If f is moreover injective and $b \in f(A_0) \cap f(A_1)$, then $b = f(a_0)$ for some $a_0 \in A_0$ and $b = f(a_1)$ for some $a_1 \in A_1$. By injectivity $a_0 = a_1 \in A_0 \cap A_1$, so $b = f(a_0)$ means that $b \in f(A_0 \cap A_1)$. Therefore if f is injective, $f(A_0) \cap f(A_1) \subseteq f(A_0 \cap A_1)$, so equality holds.

(h) Suppose $b \in f(A_0) - f(A_1)$. Then $b \in f(A_0)$ and $b \notin f(A_1)$. Respectively, this means $b = f(a_0)$ for some $a_0 \in A_0$ and $b \neq f(a_1)$ for all $a_1 \in A_1$. If $a_0 \in A_1$ then $b = f(a_0)$ is a contradiction, so $a_0 \in A_0 - A_1$. Hence $b = f(a_0)$ implies $b \in f(A_0 - A_1)$, showing that $f(A_0) - f(A_1) \subseteq f(A_0 - A_1)$.

If f is moreover injective and $b \in f(A_0 - A_1)$, then $b = f(a)$ for some $a \in A_0 - A_1$. Since $A_0 - A_1 \subseteq A_0$, we have $a \in A_0$, so $b \in f(A_0)$. Suppose there exists $a_1 \in A_1$ such that $b = f(a_1)$. Then by injectivity, $f(a) = f(a_1)$ implies $a = a_1$, but $a \notin A_1$ while $a_1 \in A_1$; a contradiction. Thus $b \neq f(a_1)$ for all $a_1 \in A$, meaning $b \notin f(A_1)$. Since $b \in f(A_0)$ and $b \notin f(A_1)$, we have $b \in f(A_0) - f(A_1)$. Therefore if f is injective, $f(A_0 - A_1) \subseteq f(A_0) - f(A_1)$, and equality holds by the previous paragraph.

3. (b) Let $\{B_\alpha\}_{\alpha\in J}$ be an arbitrary family of subsets of B.

Suppose $a \in f^{-1}\left(\bigcup_{\alpha\in J} B_\alpha\right)$. Then $f(a) \in \bigcup_{\alpha\in J} B_\alpha$; let $\alpha_0 \in J$ be such that $f(a) \in B_{\alpha_0}$. This means $a \in f^{-1}(B_{\alpha_0}) \subseteq \bigcup_{\alpha\in J} f^{-1}(B_\alpha)$, showing that $f^{-1}\left(\bigcup_{\alpha\in J} B_\alpha\right) \subseteq \bigcup_{\alpha\in J} f^{-1}(B_\alpha)$.

Conversely suppose $a \in \bigcup_{\alpha\in J} f^{-1}(B_\alpha)$. Let $\alpha_0 \in J$ be such that $a \in f^{-1}(B_{\alpha_0})$. Then $f(a) \in B_{\alpha_0} \subseteq \bigcup_{\alpha\in J} B_\alpha$. This means $a \in f^{-1}\left(\bigcup_{\alpha\in J} B_\alpha\right)$. Therefore $\bigcup_{\alpha\in J} f^{-1}(B_\alpha) \subseteq f^{-1}\left(\bigcup_{\alpha\in J} B_\alpha\right)$, so equality holds by the previous paragraph.

(c) Let $\{B_\alpha\}_{\alpha\in J}$ be an arbitrary family of subsets of B.

$a \in f^{-1}\left(\bigcap_{\alpha\in J} B_\alpha\right)$ is equivalent to $f(a) \in \bigcap_{\alpha\in J} B_\alpha$, which is equivalent to $f(a) \in B_\alpha$ for all $\alpha \in J$. By definition, this occurs if and only if $a \in f^{-1}(B_\alpha)$ for all $\alpha \in J$, and equivalently $a \in \bigcap_{\alpha\in J} f^{-1}(B_\alpha)$. Thus $f^{-1}\left(\bigcap_{\alpha\in J} B_\alpha\right) = \bigcap_{\alpha\in J} f^{-1}(B_\alpha)$.

(f) Let $\{A_\alpha\}_{\alpha\in J}$ be an arbitrary family of subsets of A.

Suppose $b \in f\left(\bigcup_{\alpha\in J} A_\alpha\right)$. Then $b = f(a)$ for some $a \in \bigcup_{\alpha\in J} A_\alpha$. Let $\alpha_0 \in J$ be such that $a \in A_{\alpha_0}$. Then $b = f(a)$ means $b \in f(A_{\alpha_0}) \subseteq \bigcup_{\alpha\in J} f(A_\alpha)$. Therefore $f\left(\bigcup_{\alpha\in J} A_\alpha\right) \subseteq \bigcup_{\alpha\in J} f(A_\alpha)$.

Conversely, if $b \in \bigcup_{\alpha\in J} f(A_\alpha)$, then let $\alpha_0 \in J$ be such that $b \in f(A_{\alpha_0})$. This means $b = f(a)$ for some $a \in A_{\alpha_0}$. Clearly $A_{\alpha_0} \subseteq \bigcup_{\alpha\in J} A_\alpha$, so $a \in \bigcup_{\alpha\in J} A_\alpha$. Now $b = f(a)$ implies $b \in f\left(\bigcup_{\alpha\in J} A_\alpha\right)$. Thus $\bigcup_{\alpha\in J} f(A_\alpha) \subseteq f\left(\bigcup_{\alpha\in J} A_\alpha\right)$, so equality holds.

(g) Let $\{A_\alpha\}_{\alpha\in J}$ be an arbitrary family of subsets of A.

Suppose $b \in f\left(\bigcap_{\alpha\in J} A_\alpha\right)$. Then $b = f(a)$ for some $a \in \bigcap_{\alpha\in J} A_\alpha$. In particular, $a \in A_\alpha$ for each $\alpha \in J$, so $b = f(a)$ shows that $b \in f(A_\alpha)$ for every $\alpha \in J$. Thus $b \in \bigcap_{\alpha\in J} f(A_\alpha)$, so $f\left(\bigcap_{\alpha\in J} A_\alpha\right) \subseteq \bigcap_{\alpha\in J} f(A_\alpha)$.

Suppose f is moreover injective and $b \in \bigcap_{\alpha\in J} f(A_\alpha)$. Then for every $\alpha \in J$, $b \in f(A_\alpha)$, meaning there exists $a_\alpha \in A_\alpha$ such that $b = f(a_\alpha)$. By injectivity, all these a_α must be the same; that is, for any $\alpha, \beta \in J$, $f(a_\alpha) = b = f(a_\beta)$ implies $a_\alpha = a_\beta$. In particular, $a_\alpha \in \bigcap_{\alpha\in J} A_\alpha$ satisfies $b = f(a_\alpha)$, showing that $b \in f\left(\bigcap_{\alpha\in J} A_\alpha\right)$. Therefore if f is injective then $\bigcap_{\alpha\in J} f(A_\alpha) \subseteq f\left(\bigcap_{\alpha\in J} A_\alpha\right)$, so equality holds.