**Do not open this notebook until instructed.**

MAT 327 Introduction to Topology

# Term Test

University of Toronto, October 16, 2024

**Solve all 5 problems on this booklet.**
The problems are of equal weight.
You have an hour and fifty minutes to write this test.

**Notes**

- No outside material allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- Write your solution of each problem on the problem page and on the back of the problem page. If you run out of space you may continue into the scratch pages, but you **must** indicate this on the problem page or else the scratch pages will not be read.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- **In Red.** Comments added after the test took place.

**Good Luck!**

**Problem 1.** Let $X$ be a topological space and let $f, g\colon X \to \mathbb{R}$ be a pair of continuous functions.

1. Show that $\{x \in X\colon f(x) = g(x)\}$ is a closed set in $X$.

2. A subset $D \subset X$ is called *dense* if it has a non-empty intersection with every ~~closed~~ non-empty open set in $X$ (example: $D = \mathbb{Q}$ is dense in $X = \mathbb{R}$). With $f$ and $g$ as before, show that if $f$ is equal to $g$ on a dense set $D$, then $f$ is equal to $g$ everywhere.

**Tip.** Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

**Problem 2.** Show that if a topological space $X$ is Hausdorff and if $x_1, \ldots, x_n$ are distinct points in $X$, then there exist open sets $U_1, \ldots, U_n$ in $X$ such that $\forall i\, x_i \in U_i$ and such that $\forall i \neq j$, $U_i \cap U_j = \emptyset$.

**Tip.** In math exams, "show" means "prove".

**Problem 3.** Let $X$ be a metrizable space and let $A \subset X$ be some subset of $X$. Show that a point $x$ belongs to $\bar{A}$ if and only if there exists a sequence $a_i$ of points in $A$ such that $a_i \to x$.

**Tip.** "If and only if" always means that there are two things to prove.

**Problem 4.** Let $X_n$ be a sequence of non-empty topological spaces whose topology is not the trivial topology. Show that the boxes topology on $X := \prod_{n=1}^{\infty} X_n$ is strictly stronger than the cylinders topology on $X$.

**Tip.** Here and almost always, a concise yet precise solution is better than a lengthy roundabout one.

**Problem 5.** Let $X$ be a topological space, let $\sim$ be an equivalence relation on $X$, let $Y = X/\sim$ be the quotient set, and let $\pi\colon X \to Y$ be the natural projection.

1. Write the "functional" definition of the quotient topology on $Y$. Namely, list some functions or some families of functions whose range is $Y$ or whose domain is $Y$ that must be continuous relative to the quotient topology on $Y$.

2. Prove that if a quotient topology on $Y$ exists, then it is unique. (Note that you are not asked to prove that the quotient topology exists!).

**Tip.** For all problems, you may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space allocated for the solutions.
**Tip.** Once you have finished writing an exam, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.