24-327 on Tuesday September 3:

~~18-327 on Thursday September 6, hours 1-2~~: Course Introduction

GB248!

September-11-10 12:29 PM

Third year → Fundamental → No mercy →

Remember blackboard shots!

MAT 327 Introduction to Topology

DROR BAR-NATAN http://drorbn.net/~~18~~ 24-327

TAs: Brinda Venkataramani and Kai Shaikh

Today's reading: Munkres: All introductions, Chapter 1 Sections 1-8, Chapter 2 Section 12-13

* Discuss the two web pages: Front page and About

Agenda: Understand "continuity" at the highest generality!

Definition A function $f:\mathbb{R}^n\to\mathbb{R}^m$ is "continuous" if

$\forall x_0\in\mathbb{R}\ \forall\epsilon>0\ \exists\delta>0\ \forall x\in\mathbb{R}\ |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\epsilon$

Theorem $f:\mathbb{R}\to\mathbb{R}$ is continuous iff for every open set $U\subset\mathbb{R}$, $f^{-1}(U)$ is also open.

* Define $B_\epsilon(x_0)$ and open sets give some examples.

* A few words on "f^{-1}"

* Proof. $\Rightarrow$ Assume $f:\mathbb{R}^n\to\mathbb{R}^m$ is cont., $U\subset\mathbb{R}^m$ is open, $x_0\in f^{-1}(U)$. Then $f(x_0)\in U$ so pick $\epsilon>0$ s.t. $B_\epsilon(f(x_0))\subset U$ and δ s.t. $|x-x_0|<\delta\Rightarrow|f(x)-f(x_0)|<\epsilon$; this really means $f(B_\delta(x_0))\subset B_\epsilon(f(x_0))$ implying $B_\delta(x_0)\subset f^{-1}(f(B_\delta(x_0)))\subset f^{-1}(B_\epsilon(f(x_0)))\subset f^{-1}(U)$, so we found a ball about x_0 contained in $f^{-1}(U)$, so $f^{-1}(U)$ is open.

$\Leftarrow$ Assume that for every open $U\subset\mathbb{R}^m$, $f^{-1}(U)\subset\mathbb{R}^n$ is open, and let $x_0\in\mathbb{R}^n$ & $\epsilon>0$ be given.

Then $B_\epsilon(f(x_0))$ is open, so $f^{-1}(B_\epsilon(f(x_0)))$ is open

but $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ so $\exists\, \delta > 0$ s.t.

$B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$, which means that

$$|y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \epsilon$$

and we have proven the continuity of f at x_0.

$\square$

Properties of open sets:

1. $\emptyset, \mathbb{R}$ 2. $\bigcup$ 3 Finite $\bigcap$ } stated, not proven

2010 done line for hour 1

Definition 1. A topological space

done the

2. Continous function $f: X \to Y$. } hinted.

Theorem The composition of continuous functions is continuous.

Examples The discrete and trivial topologies, continuous functions $f: X_{discrete, trivial} \to \mathbb{R}$

$f: \mathbb{R} \to X_{discrete, trivial}$.

Def Homeomorphism, homeomorphic

Examples. 1. $(-\frac{\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$.

2. $(-1,1)$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$

3. $\mathbb{R}$ is homeomorphic to $(-1,1)$.

Go over "index" & "About"

Definition $\mathcal{T}_1 \supset \mathcal{T}_2$ is "$\mathcal{T}_1$ is finer / bigger / stronger than $\mathcal{T}_2$" while "$\mathcal{T}_2$ is coarser / smaller / weaker" than $\mathcal{T}_1$.

The identity is continuous iff it goes from the finer topology to the weaker one.

Claim $Id:(X,\mathcal{T}_1)\to(X,\mathcal{T}_2)$ is a homeo iff $\mathcal{T}_1=\mathcal{T}_2$.

Def A "basis for a topology on a set X" is a collection $\mathcal{B}\subset P(X)$ s.t.

not same as linalg!

1. $\forall x\in X\ \exists B\in\mathcal{B}$ s.t. $x\in B$
2. $\forall B_{1,2}\in\mathcal{B}\ \forall x\in B_1\cap B_2\ \exists B_3\in\mathcal{B}$ s.t. $x\in B_3\subset B_1\cap B_2$

Examples 1. $\{(a,b)\}$ 2. $\{[a,b)\}$

Thm Given a basis for a topology on X,

1. There exists a unique minimal topology $\mathcal{T}_\mathcal{B}$ containing $\mathcal{B}$.
2. $U\in\mathcal{T}_\mathcal{B}\iff\forall x\in U\ \exists B\in\mathcal{B}$ s.t. $x\in B\subset U$
3. $\mathcal{T}_\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$.

2010 hour 3 done line

$V \subset \mathbb{R}^n$ open $\iff \forall x \in U\ \exists \epsilon > 0\ \ B_\epsilon(x) \subset U$

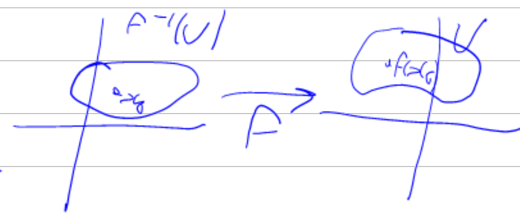
$f : \mathbb{R}^n \to \mathbb{R}^m$ cont. $\iff \forall x_0 \in \mathbb{R}^n\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}^n$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

Thm $f : \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff $\forall U \subset \mathbb{R}^m$ open $f^{-1}(U)$ is also open.

Pf $\Rightarrow$ Assume f is cont. & let $U \subset \mathbb{R}^m$ be open. board line

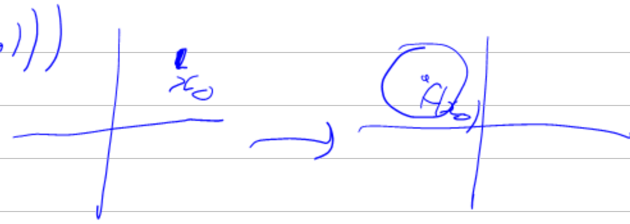
Let $x_0 \in f^{-1}(U)$, meaning that $f(x_0) \in U$. Pick $\epsilon > 0$ s.t. $B_\epsilon(f(x_0)) \subset U$ & $\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.



Then if $x \in B_\delta(x_0) \Rightarrow \ldots \Rightarrow f(x) \in B_\epsilon(f(x_0)) \subset U$

so $x \in f^{-1}(U)$.

$\Leftarrow$ Assume $f^{-1}(U)$ is open for any open U & let $x_0 \in \mathbb{R}^n$ & $\epsilon > 0$ be given. Then $f^{-1}(B_\epsilon(f(x_0)))$ is open.....



Go over home page & About docs

Properties of open sets: 1. $\emptyset$, $\mathbb{R}^n$ 2. Unions 3. Finite inter.

Def 1. Topological space, topology

2. $f : X \to Y$ cont.

Thm The composition of cont. fnctns is cont.

Examples The discrete & trivial topologies

cont. functions $f: X_{disc, trivial} \longrightarrow \mathbb{R}$

$f: \mathbb{R} \longrightarrow X_{disc, trivial}$.

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3. $\mathbb{R}$ is homeo to $(-1, 1)$

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not same as linalg!

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3. $\mathcal{T}_{\mathcal{B}}$ is the collection of all unions of elements of $\mathcal{B}$.

2010 hour 3 done line

24-327 Topology on September 10, Hour 4: Basis for a Topology.
Read Along: Munkres sections 12-14, preread 15-16.
HW2 on web by midnight!

Find BBS volunteers!
Update on waiting list.

Def: "Topology": $\mathcal{T} \subset P(X)$ s.t. 1. $\emptyset, X \in \mathcal{T}$

2. Arbitrary unions 3. Finite intersections

$\mathcal{T}_{std}$ (only $\mathbb{R}^n$) X_{triv} X_{disc} X_{fc}

$F: X \to Y$ cont: $\forall U \in \mathcal{T}_Y,\ F^{-1}(U) \in \mathcal{T}_X$

Thm A composition of cont. fncts is cont.

Intuti: On X, $\mathcal{T}_1 \supset \mathcal{T}_2$ is "$\mathcal{T}_1$ is finer / bigger / stronger than $\mathcal{T}_2$"

is "$\mathcal{T}_2$ is coarser / smaller / weaker than $\mathcal{T}_1$" $\Leftrightarrow$ $Id: (X, \mathcal{T}_{\cancel{X}}) \to (X, \mathcal{T}_X)$ is cont.

Def Homeomorphism
$h: X \to Y$, homeomorphic

$(-1,1) \sim (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$
transitivity!

Q. when is $Id: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ a homeo?

Def A "basis for a topology on X" is $\mathcal{B} \subset P(X)$ s.t.

1. $\forall x \in X\ \exists B \in \mathcal{B}$ s.t. $x \in B$ [X is a union]

2. $\forall B_{1,2} \in X\ \forall x \in B_1 \cap B_2\ \exists B \in \mathcal{B}\ x \in B$ [$B_1 \cap B_2$ is a union]

Examples 1. $\{B_r(x)\}$ 2. $\{(a,b)\}$ 3. $\{[a,b)\}$ "the lower limit topology"

Thm $\mathcal{T}_{\mathcal{B}} = \{U \subset X : \forall x \in U\ \exists B \in \mathcal{B}\ x \in B\}$
$= \{U \subset X : U$ is a union of basic sets$\}$

is a topology on X

done line

Thm $\mathcal{T}_{\mathcal{B}}$ is the unique minimal topology on X that contains $\mathcal{B}$

Thm. It is enough to verify continuity on basic sets

24-327 Topology on Thursday September 12, Hours 5-6: The order topology, induced topologies.
Read Along: Munkres sections 15-16, Preread 17.

Basis for a topology: $\mathcal{B} \subset P(X)$: 1. $\forall x \in X\ \exists B \in \mathcal{B}$ s.t. $x \in B$

2. $\forall B_{1,2} \in \mathcal{B}\ \forall x \in B_1 \cap B_2\ \exists B \in \mathcal{B}\quad x \in B \subset B_1 \cap B_2$

<u>Thm</u> $\mathcal{T}_{\mathcal{B}} = \{U \subset X : \forall x \in U\ \exists B \in \mathcal{B}\quad x \in B\}$

= All unions of sets in $\mathcal{B}$

is a topology on X

"The topology generated by $\mathcal{B}$"
"$\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$"

<u>Pf</u> ...

<u>Thm</u> $\mathcal{T}_{\mathcal{B}}$ is the unique minimal topology on X that contains $\mathcal{B}$.

<u>Remark</u> It is enough to verify continuity on basic sets.

The order topology: A "complete order" (or "simple order" or "linear order") on X is a relation on X s.t.

1. For any $x, y \in X$ exactly one of $x = y$, $x < y$, $y < x$ holds.
2. If $x < y$ & $y < z$ then $x < z$.

<u>Examples</u> 1. $(\mathbb{R}, <)$; $(\mathbb{Q}, <)$

2. English ~~words~~ strings in dictionary order: ton < top < topology
3. $\{0,1\} \times \mathbb{N}$ in dict. order.
4. $\mathbb{R} \times \mathbb{R}$ in dict. order.

<u>Def</u> The "order topology $\mathcal{T}_<$" on an ordered set X is defined by

$$\mathcal{B}_< = \{(a,b) : a < b\}$$
$$\cup \{[a_0, b) : a_0 \text{ is "minimal" in } X\}$$
$$\cup \{(a, b_0] : b_0 \text{ is "maximal" in } X\}$$

"The product topology"

Given X, Y topological spaces, we seek a topology on $X \times Y$ s.t.

1. $$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ & Y \end{tikzcd}$$ are cont.

2. $f, g : Z \to X, Y$ cont. $\Rightarrow$ $f \times g : Z \to X \times Y$ is cont.

~~switch to "constructive".~~

<u>Thm</u> Such a topology exists and is unique.

<u>Claim</u> $X \cong X \times \{y_0\}$ & $Y \cong \{x_0\} \times Y$.

The subspace Topology. Given a T.s. X and a subset $Y \subset X$, we seek a topology on Y s.t.

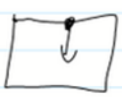
~~switch to "constructive".~~

1. $i_Y : Y \hookrightarrow X$ is cont.
2. Given $f : Z \to Y \xhookrightarrow{i_Y} X$, if $i_Y \circ f$ is cont., then so is f.

<u>Thm</u> such a topology exists and is unique.

Compatibilities. $\underbrace{\text{Sub \& Sub}}_{\text{prove}}$; $\underbrace{\text{Sub \& product}}_{\text{Leave as HW}}$; sub & order (in the convex case)

<u>Example</u> The I^2_{dict} is different from $I^2_{in} \subset \mathbb{R}^2_{dict}$.



not open

open

24-327 Topology on Tuesday September 17, Hour 7: Induced topologies, closed sets.
Read Along: Munkres sections 17-18, preread 19-20.

Given $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, seek $(X \times Y, \mathcal{T}_{X\times Y})$ s.t.

1. $\pi_X : X\times Y \to X$ & $\pi_Y : X \times Y \to Y$ are cont.

2. If $h : Z \to X \times Y$ & $\pi_X \circ h$ & $\pi_Y \circ h$ are conti,

   then h is cont.

Thm Such a topology exits & is unique; $\mathcal{B}_{X\times Y} = \{U \times V : {U \in \mathcal{T}_X \atop V \in \mathcal{T}_Y}\}$

board line.

Claim $X \sim X \times \{y_0\}$ & $Y \sim \{x_0\} \times Y$

The subspace Topology Given $(X, \mathcal{T}_X)$ and $Y \subset X$, seek a topology on Y s.t. 1. $i_Y : Y \hookrightarrow X$ is cont.

2. Given $f : Z \to Y$, if $i_Y \circ f$ is cont., the so is f

Thm Such a topology exists and is unique.

Examples 1. $[0,1]$ is open in $[0,1]$

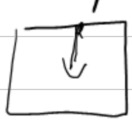
2. $[0,1)$ is open in $[0,1]$.

3. "open in open is open"

Compatibilities 1. Sub & sub (now) 2. Sub & product (HW) 3. bases & These

4. Sub & order, convex case.

Example I^2_{dict} is different from $I^2_{ind} \subset \mathbb{R}^2_{dict}$



not open open

done line.

Closed sets Def.

3 basic properties

f cont $\iff f^{-1}(\text{closed})$ is closed.

closed in a subspace

closed in closed is closed

Closure & interior, $\overline{A}$ & $A^{\circ} = \text{int} A$

Proposition $x \in \overline{A}$ iff every (basic) nbd of x intersects A.

Def Limit pt: $x \in A' \iff x \in \overline{A \setminus \{x\}}$ iff every nbd of x contains a pt of A other that x itself.

Thm $\overline{A} = A \cup A'$

Office hours moving to Tuesdays 9:30-10:30.

Closed sets Def.

3 basic properties

f cont $\Leftrightarrow f^{-1}(\text{closed})$ is closed

So we could have used closed sets as the foundation of "topology".

Example We'll define by induction sets C_n that are unions of closed intervals: $C_0 = [0,1]$. C_{n+1}: remove the open middle $1/3$ from every interval in C_n

Finally $C := \bigcap_{n=0}^{\infty} C_n$

Exercises: 1. C has "length" 0

2. C is uncountable & has many clopen subsets.

3. $\exists$ cont $f : [0,1] \to [0,1]$ $f(0)=0$ $f(1)=1$, $f'(x) = 0$ $\forall x \notin C$.

Closed in a subspace; Closed in closed is closed

Closure & interior, $\bar{A} = \mathrm{cl}_X A$ & $\mathring{A} = \mathrm{int}_X A$

Claim A open $\Leftrightarrow A = \mathrm{int} A$; A closed $\Leftrightarrow A = \mathrm{cl} A$.

Exercise $(\mathring{A})^c = \overline{(A^c)}$. (14 sets ....)

Proposition $x \in \bar{A}$ iff every (basic) nbd of x intersects A.

Def Limit pt: $x \in A' \Leftrightarrow x \in \overline{A \setminus \{x\}}$ iff every nbd of x contains a pt of A other that x itself.

Thm $\bar{A} = A \cup A'$

done line

Hausdorff spaces. Definition

Claim In a T_2 space X

1. Points are closed.
2. $x \in A'$ iff every nbd of x contains infinitely many pts of A.
3. A sequence converges to at most one limit.

Claim Products of T_2 are T_2, subspaces of T_2 are T_2

Continuous functions. TFAE for $f: X \to Y$.

1. f is cont.
2. $f(\overline{A}) \subset \overline{f(A)}$
3. For every closed B, $f^{-1}(B)$ is closed.
4. For every $x \in X$ & nbd V of $f(x)$, $\exists$ nbd U of x s.t. $f(U) \subset V$.

Pf $1 \Rightarrow 2$ $x \in \overline{A}$. Take a nbd U of $f(x)$. Then $f^{-1}(U) \cap A \neq \emptyset$, so $U \cap f(A) \neq \emptyset$ so $f(x) \in \overline{f(A)}$

$2 \Rightarrow 3$ Let $A = f^{-1}(B)$. Then $f(\overline{A}) \subset \overline{f(f^{-1}(B))} \subset \overline{B} = B$ so $\overline{A} \subset A$ so A is closed.

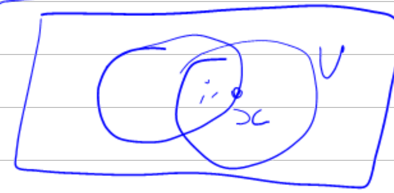
$3 \Leftrightarrow 1$, $4 \Leftrightarrow 1$ done.

24-327 Topology on Tuesday September 24, Hour 10: T2 spaces, continuous functions..
Read Along: Munkres sections 17-18, preread 19-20.

Def $A' := \{x \in X : x \in \overline{A - \{x\}}\} = \{\text{limit points of } A\}$

Theorem $x \in A' \Rightarrow$ Every nbd of x contains ∞-many elements of A

Proof



Do T_1 then...

board line

Hausdorff spaces. Definition

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done

Pf $1 \Rightarrow 2$ $x \in \overline{A}$. Take a nbd U of $f(x)$. Then $f^{-1}(U) \cap A \neq \emptyset$, so $U \cap f(A) \neq \emptyset$ so $f(x) \in \overline{f(A)}$

$2 \Rightarrow 3$ Let $A = f^{-1}(B)$. Then $f(\overline{A}) \subset \overline{f(f^{-1}(B))} \subset \overline{B} = B$ so $\overline{A} \subset A$ so A is closed.

done

$3 \Leftrightarrow 1$, $4 \Leftrightarrow 1$ done.

24-327 Topology on Thursday September 26, Hours 11-12: Continuous functions, infinite products.
Read Along: Munkres sections 18-19, preread 20-21.

∞ many prisoners get a day to (chat & prepare). Then they are set in a circle wearing b/w hats, and they each have to guess the colour of their hat, with no communications allowed. If all but finitely many of them guess right, they all survive. What do they do?

TFAE for $f: X \to Y$. 1. f is cont. 2. $f(\bar{A}) \subset \overline{f(A)}$

3. For every closed B, $f^{-1}(B)$ is closed.

4. For every $x \in X$ & nbd V of $f(x)$, $\exists$ nbd U of x s.t. $f(U) \subset V$.

$3 \Leftrightarrow 1$, $4 \Leftrightarrow 1$ done.

<u>Pf</u> $1 \Rightarrow 2$ $x \in \bar{A}$. Take a nbd U of $f(x)$. Then

$f^{-1}(U) \cap A \neq \emptyset$, so $U \cap f(A) \neq \emptyset$ so $f(x) \in \overline{f(A)}$

$2 \Rightarrow 3$ Let $A = f^{-1}(B)$. Then $f(\bar{A}) \subset \overline{f(f^{-1}(B))} \subset \bar{B} = B$

so $\bar{A} \subset A$ so A is closed.

<u>The Product Topology</u> on $X = \prod_{\alpha \in I} X_\alpha = \{x : I \to \bigcup X_\alpha : x_\alpha = x(\alpha) \in X_\alpha\}$

The Axiom of Choice: If $\forall \alpha\ X_\alpha \neq \emptyset$, this is non-empty.

Examples 1. Is $\prod_{\emptyset \neq A \subset \mathbb{R}} A$ non-empty?

2. $X = \mathbb{R}^{\mathbb{N}} = \{$sequences of real numbers$\}$

<u>Claim</u> $\exists h : X \to X$ s.t. 1. $h(a)$ depends only on the tail of a 2. a & $h(a)$ have the same tail.

Topologies on $X = \prod X_\alpha$

Generalize the construction

$\mathcal{B} = \{\prod U_\alpha : U_\alpha \subset X_\alpha \text{ is open}\}$

"The box topology"

Generalize the requirements:

1. $\pi_\alpha : X \to X_\alpha$ cont.
2. If $f : Z \to X$ is s.t. $\forall \alpha$ $f_\alpha = \pi_\alpha \circ f$ is cont., then f is cont.

$\mathcal{B} = \{\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \pi_{\alpha_n}^{-1}(U_{\alpha_n}) : \alpha_1 \dots \alpha_n \in I,\ U_{\alpha_i} \subset X_{\alpha_i} \text{ is open}$

$= \{\prod U_\alpha : U_\alpha \subset X_\alpha$ is open, for all but finitely many α, $U_\alpha = X_\alpha\}$

"The cylinders topology"
"The product topology"

For finite I, these two are the same.

Example $\mathbb{R} \to \mathbb{R}^n$ by $t \mapsto (t, t, t \dots)$ is cont. in Cyl but not in box, so box is strictly finer than cyl.

In both topologies

1. If $A_\alpha \subset X_\alpha$, the topology on $\prod A_\alpha$ as a subspace is the same as as a product.
2. If $\forall \alpha$ X_α is T_2, the so is $\prod X_\alpha$

done fine

3. $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ <u>PF</u> $x \in \overline{\prod A_\alpha} \iff$

$\forall \{U_\alpha\}_{\alpha \in I}$ $\forall \alpha$ U_α a nbd of x_α $\prod U_\alpha \cap \prod A_\alpha \neq \emptyset \iff \forall$ such U_α $\forall \alpha$ $U_\alpha \cap A_\alpha \neq \emptyset \iff \forall \alpha$ $x_\alpha \in \overline{A_\alpha}$

$\iff x \in \prod \overline{A_\alpha}$

Topologies on $\prod_\alpha X_\alpha$:

1. Generalize the basis: $\mathcal{B}_{box} = \{\prod U_\alpha : U_\alpha \subset X_\alpha \text{ open}\}$ of $\mathcal{T}_{box}$

2. Generalize the requirements ....

$$\mathcal{B}_{cyl} = \left\{\prod U_\alpha : \begin{matrix} U_\alpha \subset X_\alpha \text{ open} \\ U_\alpha = X_\alpha \text{ almost always}\end{matrix}\right\} \text{ of } \mathcal{T}_{cyl}$$

$\mathbb{R} \ni t \longmapsto (t,t,t,\dots) \in \mathbb{R}^{\mathbb{N}}$ cont. in cyl, not cont. in box. $\mathcal{T}_{box} \supsetneq \mathcal{T}_{cyl}$

Yet in both 1. subspaces behave 2. Hausdorff behaves.

3. $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ PF $x \in \overline{\prod A_\alpha} \iff$

$\forall \{U_\alpha\}_{\alpha \in I}$ $\prod U_\alpha \cap \prod A_\alpha \neq 0 \iff \forall$ such U_α $\forall \alpha\ U_\alpha \cap A_\alpha \neq \emptyset \iff \forall \alpha\ x_\alpha \in \overline{A_\alpha}$
$\forall \alpha\ U_\alpha$ a nbd of x_α
$\iff x \in \prod \overline{A_\alpha}$

Metrics & The Metric Topology

* Defs $d: X \times X \to \mathbb{R}_{\geq 0}$ s.t. 1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. triangle ineq. ... $\mathcal{B} = \{B_r(x)\}$ ... $\mathcal{T}$

* T_2

done line

* $cl = seq\text{-}cl.$

Examples 1. $\mathbb{R}^{\mathbb{N}}_{box}$ is not metrizable (no seq of positive seqs converges to $\bar{0}$)

2. $\mathbb{R}^{\mathbb{R}}_{cyl}$ is not metrizable

$A = \{f : \mathbb{R} \to \mathbb{R},\ f = 0 \text{ almost always}\}$ $\quad \mathbb{1} \in cl(A)$
$\mathbb{1} \notin seqcl(A)$

$\xrightarrow[1]{b/w} \xrightarrow[2]{b/w} \xrightarrow[3]{b/w} \cdots \xrightarrow[n]{b/w}$?

Solution sets!

$\prod X_\alpha$: If $\forall \alpha\ X_\alpha \neq \emptyset$ & for infinitely many α, $\tau_{X_\alpha} \neq \{\emptyset, X_\alpha\}$, the box topology is strictly stronger than the cylinder topology.

Metric: $d : X \times X \to \mathbb{R}_{\geq 0}$

1. $d(x,y) = 0 \Leftrightarrow x = y$ 2. $d(x,y) = d(y,x)$

3. $d(x,z) \leq d(x,y) + d(y,z)$ $\rightsquigarrow$ Balls, basis, topology "The metric topology"

"metrizable"   always T_2.

cl = seq-cl.

Examples 1. $\mathbb{R}^{\mathbb{N}}_{box}$ is not metrizable (no seq of positive seqs converges to $\bar{0}$)

2. $\mathbb{R}^{\mathbb{R}}_{cyl}$ is not metrizable

$A = \{f : \mathbb{R} \to \mathbb{R},\ f = 0 \text{ almost always}\}$ $1 \in cl(A)$
$1 \notin \text{seq-}cl(A)$

Thm A countable product of metrizable spaces is metrizable [In the product topology i.e., in the cylinder topology]

Lemma If d is a metric, then so is $\bar{d}(x,y) := \min(1, d(x,y))$, and $\bar{d}$ defines the same topology as d.

$\bar{d}$ metric:

if $d_1 + d_2 \leq 1$ easy
if $d_1 + d_2 > 1$ also easy.

Given (X_n, d_n) w/ d_n bndd by 1

set $d(x,y) := \sup_n \frac{1}{n} d_n(x_n, y_n)$

1. This is a metric!
2. The corresponding topology is the product topology.

Aside: The uniform topology:

$$d_u(x,y) := \sup_n d_n(x_n, y_n)$$

done line

$$\sup(a_n) + \sup(b_n) \geq \sup(a_n + b_n)$$

Quotient spaces. <u>Def</u> An equivalence relation on a set X is a relation $\sim$ on X s.t.

1. Reflexive 2. Symmetric 3. Transitive.

If $x \in X$, $[x] := \{y : y \sim x\} \ni x$
"the equiv class of x"

Equivalence classes are either equal or disjoint, and they cover X.

$X/\sim \; := \{\text{All equiv. classes in } X\}$ "The quotient map" $\pi : X \to X/\sim$ surjective.

<u>Claim</u> Given $p : X \to Y$ surjection, $\exists \sim$ on X s.t. there is a bijection $\alpha : Y \to X/\sim$ making the diagram on the left commutative

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ Y \arrow[rr, "\alpha"'] & & X/\sim \end{tikzcd}$$

Given a surjection $\pi : X \to Y$, or a quotient map $\pi : X \to X/\sim$, we seek a topology on $X/\sim$

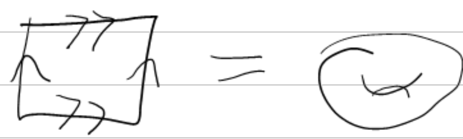
such that:

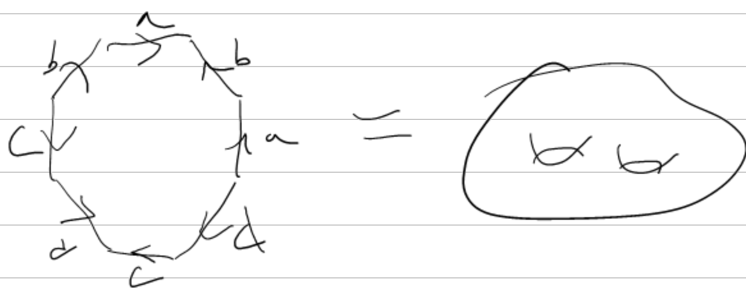
1. $\pi: X \to X/\sim$ is cont.
2. If $f: X/\sim \to Z$ is s.t. $f \circ \pi$ is cont., then f is cont.

<u>Thm</u> Such a topology exists and is unique. In fact,

$$\mathcal{T}_Y = \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}_X\}$$

<u>Examples</u> 1. $S^1 = [0,1]/_{0 \sim 1} = \mathbb{R}/\mathbb{Z}$

2. 

3. 

4. $M_{2\times 2}(\mathbb{R})/\text{conjugation}$ is not even T_2!

Indeed,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix} \not\sim \begin{pmatrix}0&1\\0&0\end{pmatrix} \text{ yet } \left[\begin{pmatrix}0&1\\0&0\end{pmatrix}\right] \supset \left\{\begin{pmatrix}0&\epsilon\\0&0\end{pmatrix} : \epsilon > 0\right\}$$

$$\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\lambda^{-1}&0\\0&\lambda\end{pmatrix} = \begin{pmatrix}0&\lambda^2\\0&0\end{pmatrix}$$

24-327 Topology on Tuesday October 8, Hours 16: Metric spaces and products, quotient spaces..
Read Along: Munkres sections 19-22, preread 23-24.
No corridor question after class today - I'll have to run to the dentist.
TT info: A week ahead.
You walk due east starting from Toronto (latitude 43.65) and going on a straight line ("geodesic").
How many km until you'll hit the equator?

Thm. $\forall n \in \mathbb{N}$ X_n is metrizable $\Longrightarrow \prod X_n$ is metrizable

Given (X_n, d_n) w/ d_n bndd by 1

set $d(x,y) := \sup_n \frac{1}{n} d_n(x_n, y_n)$

Aside: The uniform topology:

$d_u(x,y) := \sup_n d_n(x_n, y_n)$ (assuming bndness)

$\sup(a_n) + \sup(b_n) \geq \sup(a_n + b_n)$

1. This is a metric!
2. The corresponding topology is the product topology.

Quotient spaces. Def An equivalence relation on a set X is a relation $\sim$ on X s.t.

1. Reflexive 2. Symmetric 3. Transitive.

If $x \in X$, $[x] := \{y : y \sim x\} \ni x$
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Equivalence classes are either equal or disjoint, and they cover X.

$X/\sim := \{$All equiv. classes in $X\}$

"The quotient map" $\pi : X \to X/\sim$ surjective.

Claim Given $p : X \to Y$ surjection, $\exists \sim$ on X s.t. there is a bijection $\alpha : Y \to X/\sim$ making the diagram on the left commutative

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ Y \arrow[rr, "\alpha"'] & & X/\sim \end{tikzcd}$$

Given a surjection $\pi: X \to Y$, or a quotient map $\pi: X \to X/\sim$, we seek a topology on $X/\sim$ such that:

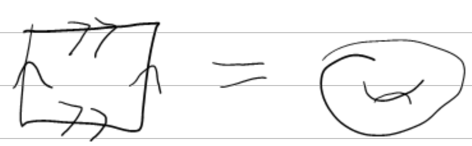
1. $\pi: X \to X/\sim$ is cont.
2. If $f: X/\sim \to Z$ is s.t. $f \circ \pi$ is cont., then f is cont.

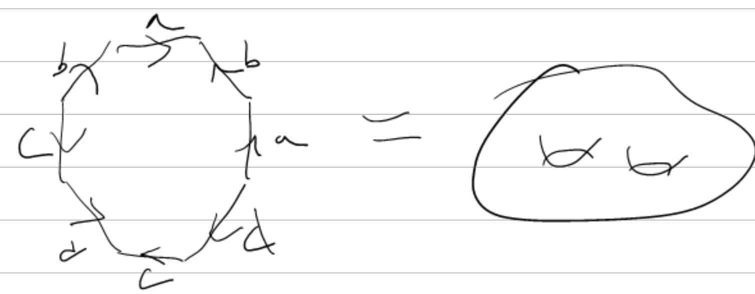
done line

Thm such a topology exists and is unique. In fact,

$$\mathcal{T}_Y = \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}_X\}$$

Examples 1. $S^1 = [0,1]/_{0\sim 1} = \mathbb{R}/\mathbb{Z}$

2. 

3. 

4 $M_{2\times 2}(\mathbb{R})/\text{conjugation}$ is not even T_2!

Indeed,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix} \not\sim \begin{pmatrix}0&1\\0&0\end{pmatrix} \text{ yet } \left[\begin{pmatrix}0&1\\0&0\end{pmatrix}\right] \supset \left\{\begin{pmatrix}0&\epsilon\\0&0\end{pmatrix} : \epsilon > 0\right\}$$

$$\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\lambda^{-1}&0\\0&\lambda\end{pmatrix} = \begin{pmatrix}0&\lambda^2\\0&0\end{pmatrix}$$

https://drorbn.net/24-327/ap/TTInfo.pdf

Dear All,

Some notes about the upcoming Term Test.

The test will take place on Wednesday October 16 at 7-9pm. If your last name falls within the interval [A,O) using the dictionary order, go to Bahen room 1180. If it's in the interval [O,Zz], go to Bahen 1220.

It will be a "closed book" exam: no books and no notes of any kind will be allowed, no cell-phones, no calculators, no devices of any kind that can display text. So only stationary will be allowed, as well as minimal hydration and snacks, and stuffed animals for joy and comfort.

The material for the test is everything that was covered in class until and including the class of Thursday October 10, unless it was clearly indicated as "extra" at the time that it was covered. Here "everything" means every bit of class material (including every single proof) and anything that was assigned as homework. It does not include material covered only in tutorials because such material differs between the different tutorial groups.

The style of the test will be "solve the following 6 questions", or maybe 5 or 8. Some of the questions will be about classroom material, like proving something that was proven in class. Some will be taken from the HW assignments, and some will be fresh.

The best way to study for the test is to make sure that you understand absolutely everything (same "everything" as before). Make a long list of definitions and lemmas and theorems, and make sure that you know every single one. Go over every homework exercise and make sure that you know how to solve it perfectly.

Neatness counts! Language counts! The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Also useful, though less, is to go over old exams. You can find a couple at https://drorbn.net/index.php?title=10-327/Term_Test and at https://www.math.toronto.edu/~drorbn/classes/18-327-Topology/TT.html (that one has a discussion about how I write exams – it remains valid).

You may also want to familiarize yourself with the physical shape of the test. See my template at https://drorbn.net/AcademicPensieve/Classes/24-327-Topology/24-327-TT-Template.pdf, but not that the number of questions may change and that the Crowdmark QR codes are not yet installed. It will be printed double sided to make a 10-16 pages booklet.

The TAs and I will hold some extra office hours before the test, as follows (a separate message with room details will be sent once the rooms are fixed):

- On Friday October 11 at 2-4pm with Brinda at Bahen 2179.
- On Tuesday October 15 at 9:30-11:30am and at 5-7pm with Dror at TBA.
- On Wednesday October 16 at 2-5pm with Kai at TBA.

Good luck!

Dror.

bring a Klein bottle & a Möbius band.

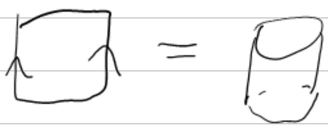
Given a surjection $\pi: X \to Y$ / a quotient map (contrast w/ injections) $\pi: X \to X/\sim$, seek a topology on $X/\sim$ such that:

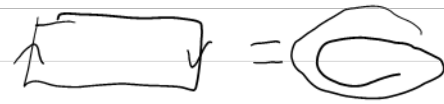
1. $\pi: X \to X/\sim$ is cont.
2. If $f: X/\sim \to Z$ is s.t. $f \circ \pi$ is cont., then f is cont.

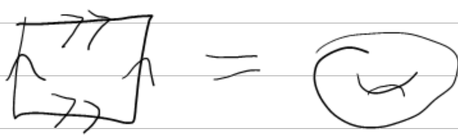
<u>Thm</u> such a topology exists and is unique.

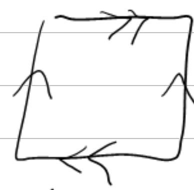
<u>Pf</u> $\mathcal{T}_Y = \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}_X\}$

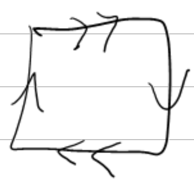
<u>Examples</u> 1. $S^1 = [0,1]/_{0\sim 1} = \mathbb{R}/\mathbb{Z}$

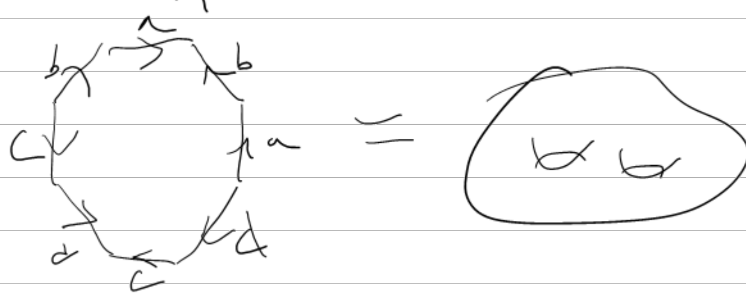
2. 

3. 

4. 

5.  Klein bottle

6.  $\mathbb{RP}^2$

7. 

8. "Vector = magnitude + direction"

$V \xrightarrow{\|\cdot\|} \mathbb{R}$

$V \xrightarrow{\pi} \operatorname{dir}(V) := V/\begin{smallmatrix} v_1 \sim v_2 \text{ if} \\ v_1 = \alpha v_2,\ \alpha > 0\end{smallmatrix}$

<u>Q</u> What is $\operatorname{dir}(\mathbb{R}^2)$ as a set & as a top space?

<u>Ans</u> , $S^1 \cup \{z\}$, w/ weird topology.

18-327 on Thursday October 18, hours 19-20: connectedness

September-11-10 12:29 PM

Today: (Above)

TT: No comments. HW4 returned

Connectedness. Separation, connectedness, clopen sets. } should have explicitly excluded $\emptyset$ as connected.

The I.V.T. If X is connected, $f: X \to \mathbb{R}$ cont., $f(x_0)<0$, $f(x_1)>0 \Rightarrow \exists x$ s.t. $f(x)=0$.

Theorem $I=[0,1]$ is connected.

Proof. Assume $0 \in A \subset I$ is clopen. Let $G=\{x : [0,x] \subset A\}$ $g=\sup G$

1. $g>0$ 2. $g \not< 1$ 3. $1 \in G$.

Theorem. A continuous image of a connected set is connected.

Theorem. If $A_\alpha \subset X$ are connected, $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is connected.

pf of the IVT

Theorem. $A \subset \mathbb{R}$ is connected iff it is an interval, or a ray, or the whole thing. [I.e., if it is "convex"]

Thm. If X & Y are connected, then so is $X \times Y$.
(Also, if $X \times Y$ is connected & $X, Y \neq \emptyset$, then X & Y are connected)

Example. $\mathbb{R}^\omega = \begin{Bmatrix}\text{bndd}\\ \text{seqs}\end{Bmatrix} \cup \begin{Bmatrix}\text{unbndd}\\ \text{seqs.}\end{Bmatrix}$ is a box-separation.

Lemma If A is connected & $A \subset B \subset \bar{A}$, B is too.

Pf Assume C is clopen in B, $C \cap A \neq \emptyset$. Then $C \supset A$ so $cl_X C \supset \bar{A} \supset B$, so $cl_X C \cap B = B$, so $cl_B C = B$, so $C=B$.

Theorem. If $\forall \alpha$ X_α is connected, then $\prod X_\alpha$ is connected.

Def. Path-connected; Path-connected $\Rightarrow$ connected
(1. Proof from defs. } not done
2. Lemma: If X is connected, so is $f(X)$)

done line.

The topologist's sine curve

A product of path connected spaces is path-connected.

24-327 Topology on Tuesday October 15, Hour 19: Connectedness.
TT tomorrow! Make sure to read drorbn.net/24-327/ap/TTInfo.pdf! Study!
What was the hardest for you?
What was the worst explained?
Tell me something that you didn't understand at first, but then did, upon reflection.

board 1

X connected: Non-$\emptyset$, no clopens except $\emptyset$ & X.

The IVT: X connected, $f: X \to \mathbb{R}$ cont., $f(x_0)<0$, $f(x_1)>0 \Rightarrow \exists\, x$ s.t. $f(x)=0$.

Pf do!

Thm $I=[0,1]$ is connected.

Pf Assume $A \subset I$ clopen, $0 \in A$.
$G := \{g \in I : [0,x] \subset A\}$ $m := \sup G > 0$
$m \neq 1$ or else A closed $\Rightarrow m \in A$;
A open $\Rightarrow \exists\, \epsilon>0$ $(m-\epsilon, m+\epsilon) \subset A$
$\Rightarrow m + \frac{\epsilon}{2} \in G \Rightarrow m \geq m + \frac{\epsilon}{2}$.

board 2.

Now prove $1 \in G$.

Theorem. A continuous image of a connected set is connected.

Theorem. If $A_\alpha \subset X$ are connected, $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is connected.

Theorem. $A \subset \mathbb{R}$ is connected iff it is an interval, or a ray, or the whole thing. [I.e., if it is "convex"]

done line

Thm. If X & Y are connected, then so is $X \times Y$.
(Also, if $X \times Y$ is connected & $X, Y \neq \emptyset$, then X & Y are connected)

Example. $\mathbb{R}^\omega = \left\{\begin{matrix}\text{bndd}\\ \text{seqs}\end{matrix}\right\} \cup \left\{\begin{matrix}\text{unbndd}\\ \text{seqs.}\end{matrix}\right\}$ is a box-separation.

Lemma If A is connected & $A \subset B \subset \bar{A}$, B is too.

Pf Assume C is clopen in B, $C \cap A \neq \emptyset$. Then $C \supset A$ so $cl_X C \supset \bar{A} \supset B$,
so $cl_X C \cap B = B$, so $cl_B C = B$, so $C = B$.

Theorem. If $\forall \alpha$ X_α is connected, then $\prod X_\alpha$ is connected.

24-327 Topology on Thursday October 17, Hours 20-21: Connectedness path connectedness, compactness.
Read Along: 23-24. Read ahead: 26-27.

$[0,1]$ is connected, cont. image of connected is connected, union of connected w/ a non-empty intersection is connected.

Thm A subset of $\mathbb{R}$ is connected iff it is an interval or a ray or the whole thing (i.e., iff it is convex).

Thm X & Y are connected $\Leftrightarrow$ $X\times Y$ is.

Example $\mathbb{R}^{\mathbb{N}}_{box} = \{\text{bndd seqs}\} \cup \{\text{unbounded seqs}\}$ is a separation.

Lemma If $A\subset X$ is connected and $A\subset B\subset \bar{A}$, then B is too.

Pf Assume C is clopen in B, $C\cap A\neq\emptyset$. Then $C\supset A$.

So $C = cl_B C = cl_X C\cap B$
$\supset \bar{A}\cap B = B$

Lemma² If $C\subset B\subset X$ then
$cl_B C = cl_X C\cap B$

Pf ~~rhs is a closed set in B which contains C. Minimality: If F is a closed set in B which contains C, then $F = B\cap E$ where $E\subset X$ is closed. So $E\supset cl_X C$ / $\cap B$; $F\supset cl_X C\cap B$~~

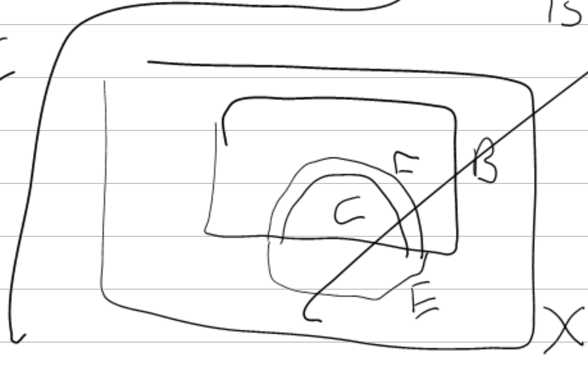
do other version.

Pf2 of Lemma²

$x\in cl_B C \Leftrightarrow (x\in B)\wedge\left(\begin{smallmatrix}\text{every nbd}\\ \text{of } x \text{ in } B\\ \text{intersects } C\end{smallmatrix}\right)$

$\Leftrightarrow (x\in B)\wedge\left(\begin{smallmatrix}\text{every nbd of}\\ x \text{ in } X \text{ intersects } C\end{smallmatrix}\right)$

$\Leftrightarrow x\in B\cap cl_X C$



Thm $\prod X_\alpha$ connected $\iff \forall \alpha\ X_\alpha$ is connected.

Def X is path-connected if $X \neq \emptyset$ & $\forall a, b \ldots$

Thm path connected $\Rightarrow$ connected.

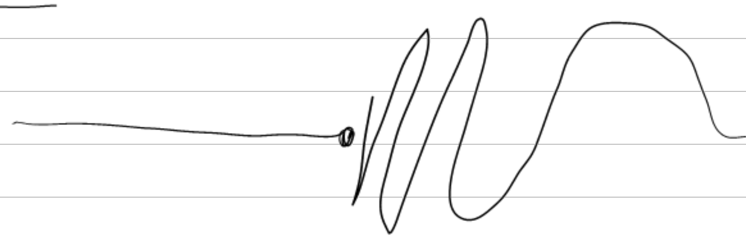
IVT is free!

$[0,1]$ is path connected (free)

Cont. image of p.c. is p.c.

$\prod X_\alpha$ path connected $\iff \forall \alpha\ X_\alpha$ is p.c.

Example



The topologist's sine curve connected but not p.c.

Riddle Find two disjoint connected subset $A, B \subset [0,1]^2$

s.t. $(0,0), (1,1) \in A$ yet $(0,1), (1,0) \in B$.

Can't w/ p.c., but that's a hard theorem.

done live

Def Cover, open cover, compact.

Thm A cont. fnctn on a cmpct space is bnd.

pf 1 Local to global.

pf 2 $X = \bigcup f^{-1}(-n, n)$

Example. Finite sets. **Thm** $[0,1]$ is cmpct.

Goal: Immitate "On a compact set, every cont. function is bounded."

Def Cover, open cover, compact.

Thm A cont. fnctn on a cmpct space is bnd.

pf 1 Local to global.

pf 2 $X = \bigcup f^{-1}(-n, n)$

Example. Finite sets. **Thm** $[0,1]$ is cmpct.

Proof Let $\mathcal{U}$ be an open cover of $[0,1]$

Let $G = \{g \in [0,1] :$ a finite subset of $\mathcal{U}$ covers $[0,g]\}$ Wish: $1 \in G$

G is non-empty & bndd, so $m = \sup G$ exists.

Steps: 1. $m > 0$ 2. $m = 1$ 3. $1 \in G$

done line

Thm A closed subset of a cmpct space is cmpct.

Thm A compact subspace of a T_2 space is closed.

Corollary: A compact T_2 space is T_3 / regular.

Corollary: A subset of $\mathbb{R}$ is cmpct iff it is closed & bndd.

24-327 Topology on Thursday October 24, Hours 23-24: Compactness.
Read Along: 26-27. TT discussion. New volunteer for blackboard shots!

Def Compact: Every open cover has a finite subcover.

Thm A cont. fnctn on a cmpct space is bndd.

Thm $[0,1]$ is cmpct.

Thm. A closed subset of a compact space is cmpct.

Thm A compact subspace of a T_2 space is closed.

Cor. A compact T_2 space is T_3 / regular.

Cor. A subset of $\mathbb{R}$ is cmpct iff it is closed & bndd.

Thm A cont. image of a cmpct set is cmpct.

Cor The maximal value thm.

Thm X, Y cmpct $\Rightarrow X \times Y$ cmpct. [and nearly vice versa]

Cor A subset of $\mathbb{R}^n$ is cmpct iff it is closed & bndd.

Def Let X, Y be metric. $f: X \to Y$ is "uniformly cont."

done till

Thm Given a cover $\mathcal{U}$ of a cmpct metric space X,

$\exists \delta_0 > 0$ s.t. $\forall x \in X \; \exists U \in \mathcal{U} \;\; B_{\delta_0}(x) \subset U$

Pf set $\Delta(x) = \sup\{\delta \leq 1 : \exists U \in \mathcal{U} \; B_\delta(x) \subset U\}$

if $d(x,y) < \epsilon$ then $\Delta(y) > \Delta(x) - \epsilon$, so $|\Delta(x) - \Delta(y)| < \epsilon$

so δ is cont. Take $\delta_0 = \min \Delta$.

Thm. X compact metric, Y metric, $f: X \to Y$ cont.

$\Rightarrow f$ is uniformly cont.

Term Test results.

Dear Students -

The results of the term test are in and will be distributed within an hour or so. They are similar to what I expected them to be, with a class average of 79 and a median grade of 87. In fact, here is the full list of marks obtained in this test: 11 26 27 35 35 37 42 43 46 48 49 51 51 51 53 54 54 55 57 61 61 62 67 68 70 71 73 75 77 78 78 79 79 80 81 82 82 83 84 84 85 85 85 85 86 87 87 88 89 89 89 90 90 90 91 91.5 92 93 93 93 94 94 94 94 94 94 94 94 94 94 94 95 95 95 95 98 99 100 100 100 100 100 100 100 100 100 100 100 100 100.

How should you read your grade?

If you got 100 you should pat yourself on your shoulder and feel good.
If you got something like 90, you're doing great. You made a few relatively minor mistakes; find out what they are and try to avoid them next time.
If you got something like 80, you're doing fine but you did miss something significant, probably more than just a minor thing. Figure out what it was and make a plan to fix the problem for next time.
If you got something like 60 you should be concerned. You are still in position to improve greatly and get an excellent grade at the end, but what you missed is quite significant and you are at the risk of finding yourself far behind. You must analyze what happened - perhaps it was a minor mishap, but more likely you misunderstood something major or something major is missing in your background. Find out what it is and try to come up with a realistic strategy to overcome the difficulty!
If you got something like 30, most likely you are not gaining much from this class and you should consider dropping it, unless you are convinced that you fully understand the cause of your difficulty (you were very sick, you really couldn't study at all for the two weeks before the exam because of some unusual circumstances, something like that) and you feel confident you have a fix for next time. If you do decide to drop the class, don't feel too bad about it - it's one of the most abstract math classes here at UofT, and it really is tough.
Note that problems with writing are problems, period. Perhaps you got a low grade but you feel you know the material enough for a high grade only you didn't write everything you know or you didn't it write well enough or the silly graders simply didn't get what you wrote (and it isn't a simple misunderstanding - see "appeals" below). If this describes you, don't underestimate your problem. If you don't process and resolve it, it is likely to recur.

Solution Sets. There will be no "official" solution set, yet students are encouraged to submit the solutions to be placed on the class's web site, in a manner similar to the solutions for the HW assignments.

Appeals. Remember! We try hard yet grading is a difficult process and mistakes always happen - solutions get misread, parts are forgotten, etc. You must read your exam and make sure that you understand how it was graded. If you disagree with anything, don't hesitate to complain! (Though first consider very carefully the possibility that the mistake is actually yours). Your first stop should be the person who graded the problem in question, and only if you can't agree with him you should appeal to Dror (within a day or two).

Brinda marked questions 1 and 3, Kai questions 2 and 5, and Dror question 4. The deadline to start the appeal process is Thursday November 7 at 5PM. Once you've started the process by talking to Dror or to one of the TAs, it ends when a final decision is made, with no deadline.

Best,

Dror.

24-327 Topology on Tuesday November 5, Hour 25: Uniform Continuity; then with great pain, "algebraic topology".
HW6 due 11:59pm, HW7 to be assigned by midnight.
Read Along: 26-27. Read Ahead: 51-55.
Our final: Mon Dec 16 2-5pm at KC Knox

Def Compact: Every open cover has a finite subcover.

Def X, Y metric. $f: X \to Y$ uniformly cont. means

$$\forall \epsilon > 0 \; \exists \delta > 0 \quad d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$$

Thm If X is compact, cont. $\Rightarrow$ unif. cont.

Thm Given a cover $\mathcal{U}$ of a cmpt metric space X

$\exists \delta_0 > 0$ s.t. $\forall x \in X \; \exists U \in \mathcal{U} \quad B_{\delta_0}(x) \subset U$

Pf set $\Delta(x) = \sup\{\delta \le 1 : \exists U \in \mathcal{U} \; B_\delta(x) \subset U\}$

if $d(x, y) < \epsilon$ then $\Delta(y) > \Delta(x) - \epsilon$, so $|\Delta(x) - \Delta(y)| < \epsilon$

so δ is cont. Take $\delta_0 = \min \Delta$.

Thm. X compact metric, Y metric, $f: X \to Y$ cont.

$\Rightarrow f$ is uniformly cont.

Tychonoff's Thm. An arbitrary product of cmpct spaces is cmpct.

$\nexists$ cont $f: D^n \to S^{n-1}$ s.t. $f|_{S^{n-1}} = Id$.

$n=1$ connectivity.

$n>1$ Algebraic topology [we'll only do $n=2$]

... A discussion of categories & functors.

Dror Bar-Natan: Classes: 2024-25: MAT 327 Introduction to Topology: **What We Miss**

Tychonoff's theorem, the Stone-Cech compactification, ultrafilters, generalized limits, miracles.

Dror Bar-Natan: Classes: 2018-19: MAT 327 Introduction to Topology:

Compactness in Metric Spaces [Munkres 28, 43, 45]

Theorem. The Following are Equivalent for a Metric X:

1. X is compact.
2. X is "limit-point-compact".
3. X is "sequentially compact".
4. X is "totally bounded" & "satisfies Lebesgue's Lemma".
5. X is totally bounded & "complete".

Quoted phrases need to be defined!

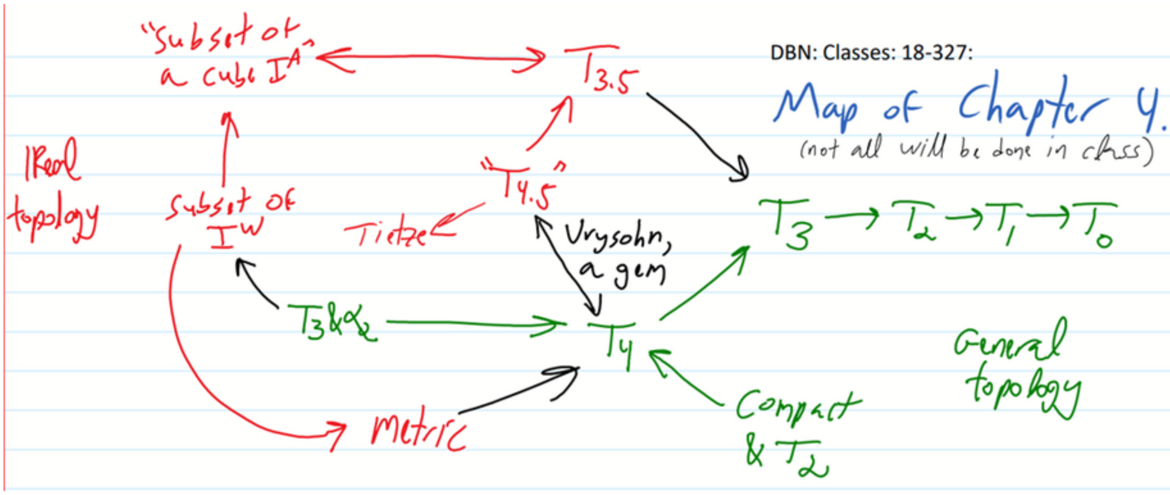
$\Updownarrow$ in $\mathbb{R}^n_{std}$ $\Updownarrow$

bounded closed

DBN: Classes: 18-327:

Map of Chapter 4.

(not all will be done in class)



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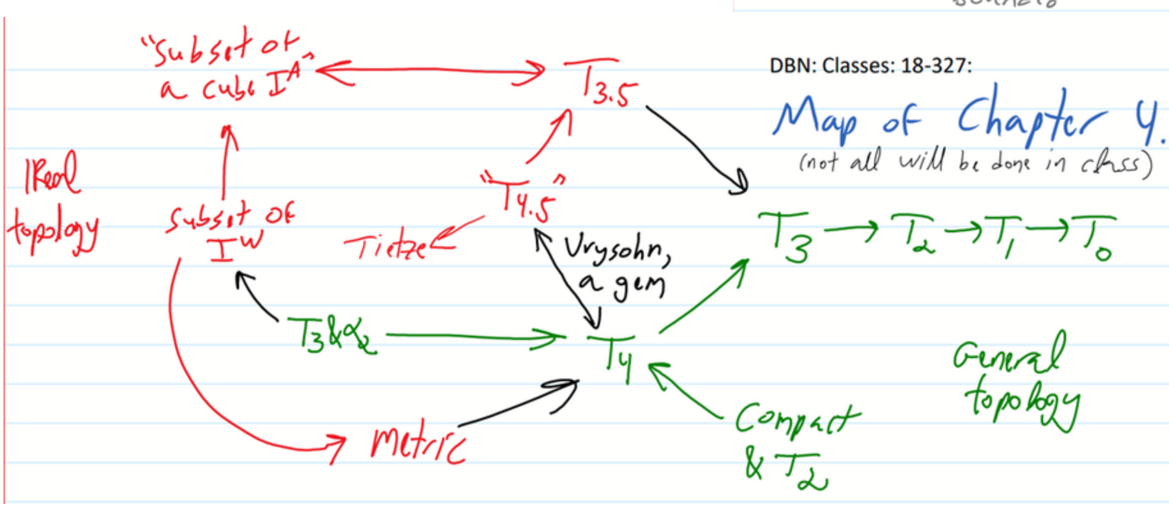
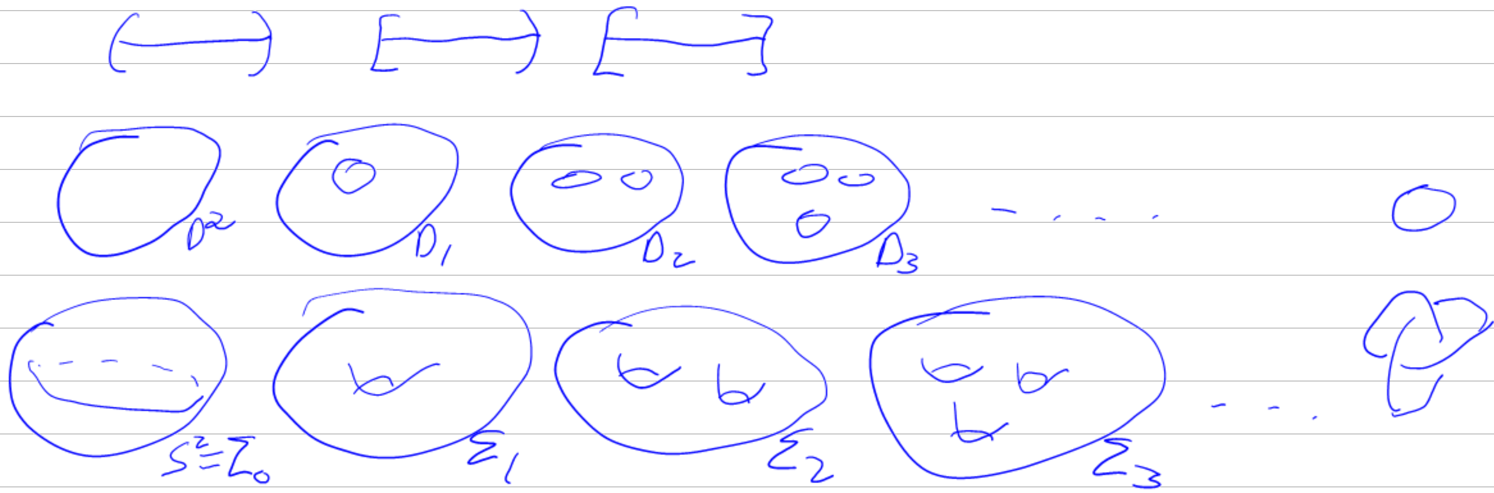


We are pathetic at telling spaces apart! Are these homeomorphic?



Stories about π_1 & the above spaces

347: Group: A set G together with a binary operation $m: G\times G \to G$ (written $(a,b)\mapsto ab$) called... a unary operation $i: G \to G$ (written $a \mapsto a^{-1}$) called... and a nullary operation $e=1\in G$ s.t.

1. Associativity 2. Unit 3. inverse.

<u>Examples</u> $\underbrace{\mathbb{Z}, \mathbb{Z}/2, \mathbb{N}, (\mathbb{R},+), (\mathbb{R}_{\neq 0}, \cdot), \mathbb{Q}, \mathbb{C},}_{\text{Abelian: } ab=ba}$ $GL_n(\mathbb{C})$

S_n, $|S_n|=n!$; $S(X)$; rotations of a cube, $|G|=24$;

The rubiks cube group $|G| = \dfrac{8!\,3^8\,12!\,2^{12}}{12}$

$=43{,}252{,}003{,}274{,}489{,}856{,}000$

Group homomorphism $\phi: G_1 \to G_2$

Examples $\mathbb{Z} \to \mathbb{R}$, $\mathbb{Z} \to \mathbb{Z}/2$, $S_n \to S_{n+1}$

$S_4 \to S_3$

Paths $\gamma(s)$; $\gamma_0 \simeq_p \gamma_1$ via H; That's an equiv. relation.

$[\gamma]$

Example Any two paths in $\mathbb{R}^n$ w/ the same endpoints are path-homotopic.

done line

$\gamma_1 * \gamma_2$ if $\gamma_1(1) = \gamma_2(0)$

Descends to $[\gamma_1] * [\gamma_2]$ if $[\gamma_1](1) = [\gamma_2](0)$

Thm Associativity, left & right identity, inverses.
e_x

Def $\pi_1(X, b)$ Example $\pi_1(\mathbb{R}^n, 0) \cong \{1\}$

Thm If X is path connected, $\pi_1(X, b_0) \cong \pi_1(X, b_1)$

Def simply-connected.

24-327 Topology on Tuesday November 12, Hour 28: The Fundamental Group \pi_1: 327->347.
HW7 due at 11:59pm, HW8 online by midnight.
Read Along: 51-55.

$\gamma_{0,1} : I_s = [0,1]_s \to X$ $\quad \gamma_0 \overset{H}{\underset{P}{\sim}} \gamma_1$ means $\exists H : I^2_{s,t} \xrightarrow{\text{cont.}} X$ s.t.

$H(0,t) = \gamma_0(0) = \gamma_1(0)$

$H(1,t) = \gamma_0(1) = \gamma_1(1)$

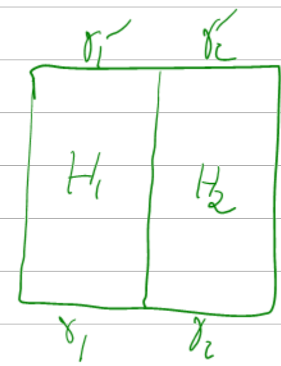
$H(s,0) = \gamma_0(s)$

$H(s,1) = \gamma_1(s)$

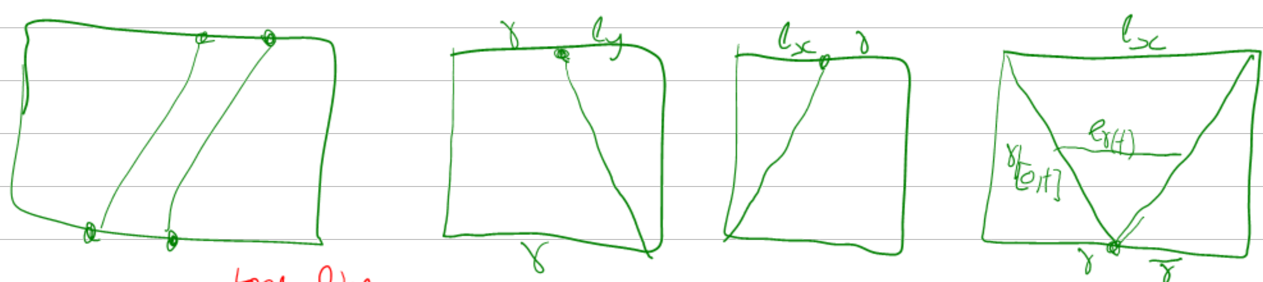
Path homotopy is an equiv. relation!

$[\gamma]$ $\quad \gamma_1 * \gamma_2$ if $\gamma_1(1) = \gamma_2(0)$

Descends to $[\gamma_1] * [\gamma_2]$ if $[\gamma_1](1) = [\gamma_2](0)$



Thm Associativity, left & right identity e_x, inverses $\bar{\gamma}$



done line

Def $\pi_1(X, b)$ $\qquad$ Example $\pi_1(\mathbb{R}^n, 0) \cong \{1\}$

Thm If X is path connected, $\pi_1(X, b_0) \cong \pi_1(X, b_1)$

Def simply-connected.

Q2 Image/PDF question HW 8 30 points

If X and Y are topological spaces, we say that two continuous functions $F_0 : X \to Y$ and $F_1 : X \to Y$ are *homotopic* if there exists a continuous function $H : X \times I \to Y$ such that for all $x \in X$, $H(x, 0) = F_0(x)$ and $H(x, 1) = F_1(x)$. In that case, we write $F_0 \sim F_1$.

(a) Prove that the relation $\sim$ is an equivalence relation.

(b) If γ is a path in X and if $F : X \to Y$ is continuous, we define a path in Y by $F_* \gamma := F \circ \gamma$. Show that if γ_0 and γ_1 are path homotopic in X then $F_* \gamma_0$ and $F_* \gamma_1$ are path homotopic in Y.

(c) Prove that if $F_i : X \to Y$ and $G_i : Y \to Z$ are continuous for $i = 0, 1$, and if $F_0 \sim F_1$ and $G_0 \sim G_1$, then $G_0 \circ F_0 \sim G_1 \circ F_1$.

Q3 Image/PDF question 20 points

Recall that $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle in $\mathbb{R}^2$.

With the same language as in the previous question, we say that a path-connected topological space X is *simply connected* if every continuous function $\lambda : S^1 \to X$ is homotopic to a constant function.

Show that a path-connected X is simply connected if and only if any two paths γ_0 and γ_1 in X that have the same endpoints (namely, $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) are path homotopic.

Q4 Image/PDF question 30 points

Recall that $S^2 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit sphere in $\mathbb{R}^3$. It is well known (and can be used freely) that S^2 with one point removed is homeomorphic to $\mathbb{R}^2$.

(a) Show that any continuous $\lambda_0 : S^1 \to S^2$ which is not surjective is homotopic to a constant function. (**Warning.** There actually exist continuous surjective maps $\lambda_0 : S^1 \to S^2$! If you don't believe, web-search "Peano Curve" and complete the missing details on your own).

(b) Show that any continuous $\lambda : S^1 \to S^2$ is homotopic to a continuous $\lambda_0 : S^1 \to S^2$ which is not surjective.

(c) With the same language as in the previous exercise, deduce that S^2 is simply connected.

$[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$ γ_1 γ_2 if $\gamma_1(1) = \gamma_2(0)$

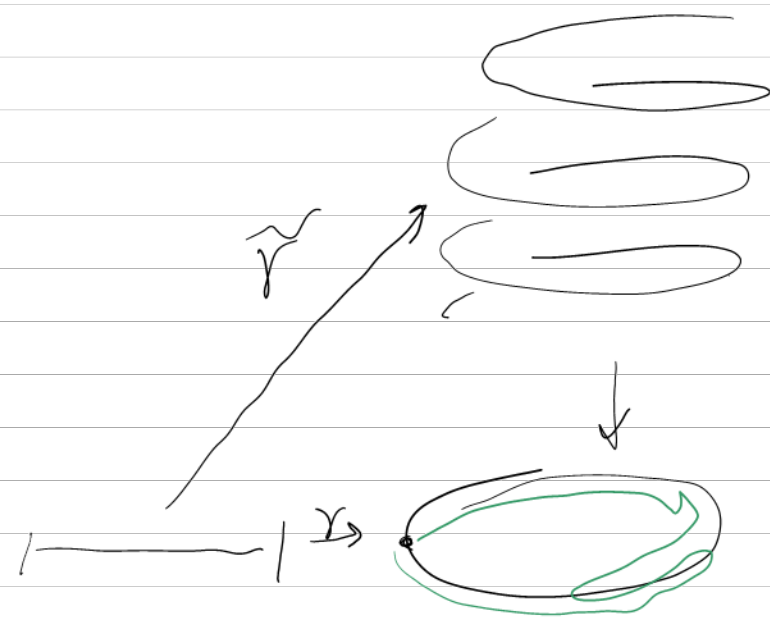
Associative, has identities & inverses.

Def $\pi_1(\underbrace{X, b}_{\text{based space}})$ Example $\pi_1(\mathbb{R}^n, 0) \cong \{1\}$

Def simply-connected.

Thm If X is path connected, $\pi_1(X, b_0) \cong \pi_1(X, b_1)$

Thm $\pi_1(S^1, 1) \cong \mathbb{Z}$



Definition $p : E \to B$ is a "covering map", and E is a "covering space of a base space B" if E is locally the product of B with a discrete set. Namely, if every $x \in B$ has a nbd U s.t. there is a discrete set D and a homeomorphism $\phi : U \times D \to p^{-1}(U)$ s.t. the diagram on the right commutes

Shopping list
1. coverings
2. Path lifting
3. Homotopy lifting

$$\begin{tikzcd} U \times D \arrow[r, "\phi"] \arrow[d, "\pi_1"] & p^{-1}(U) \arrow[d, "p"] \\ U \arrow[r, hook, "\iota"] & B \end{tikzcd}$$

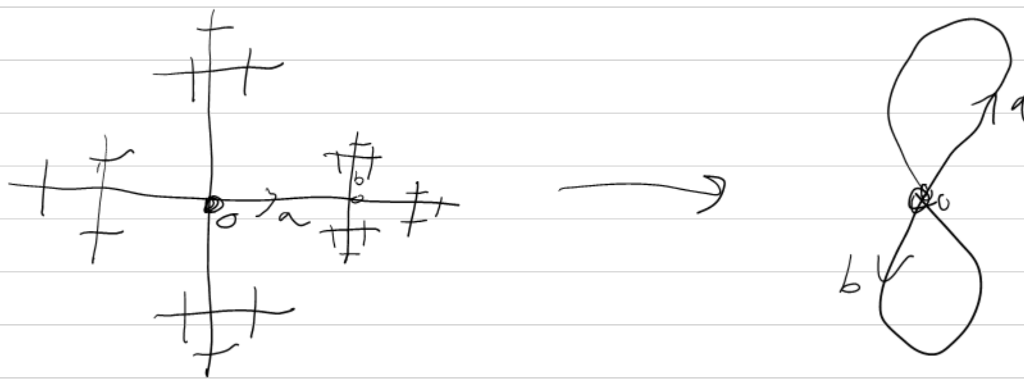
Examples 1. $B \to B$; $B \times D \to B$

2. $p: \mathbb{R} \to S^1$ via $t \mapsto e^{2\pi i t}$

3. $S^1 \to S^1$ via $z \mapsto z^3$

4. Parking garage $\longrightarrow$ basement by dropping down

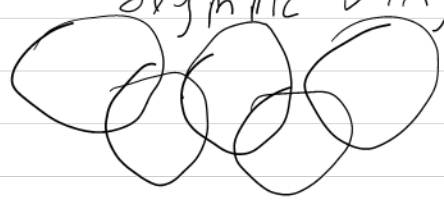
5. Mexican cross $\longrightarrow$ Chinese lucky space



mistake here

6. Long quipo $\longrightarrow$ 8

7. olympic rings

 $\longrightarrow$ 8

done line

Lemma If (X, x_0) is a connected based space, if $(E, e_0) \xrightarrow{p} (B, b_0)$ is a covering, if $\psi: (X, x_0) \to (B, b_0)$ is small, then it has a unique lift $\tilde{\psi}: (X, x_0) \to (E, e_0)$

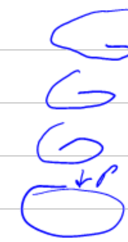
24-327 Topology on Tuesday November 19, Hour 31: The Fundamental Group \pi_1: 327->347.
HW 8 is due at 11:59, HW9 online by midnight.
Read Along: 51-55.

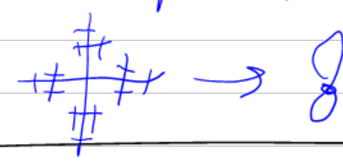
$\pi_1(X, x_0) = \left\{ [\gamma] ; \begin{matrix} \gamma : [0,1] \to X \\ \gamma(0) = \gamma(1) = x_0 \end{matrix} \right\}$ a group

Thm: $\pi_1(S^1, 1) \cong \mathbb{Z}$

$\underbrace{f : (X, x_0) \to (Y, y_0)}_{\text{based spaces}}$ means $f : X \to Y$ (cont.) & $f(x_0) = y_0$

$p : E \to B$ a covering means B can be covered by open sets $\{U_\alpha\}$ s.t. $\forall \alpha$ $p^{-1}(U_\alpha)$ is a disjoint union of open sets s.t. p is a homeomorphism of each of them with U_α.





<u>Lemma</u> If (X, x_0) is a connected based space, if $(E, e_0) \xrightarrow{p} (B, b_0)$ is a covering, if $\psi : (X, x_0) \to (B, b_0)$ is small, then it has a unique lift $\tilde{\psi} : (X, x_0) \to (E, e_0)$

<u>Lemma</u> (The path lifting property) If $p : (E, e_0) \to (B, b_0)$ is a covering & $\gamma : (I, 0) \to (B, b_0)$, then γ has a unique lift $\tilde{\gamma} : (I, 0) \to (E, e_0)$ s.t. $p \circ \tilde{\gamma} = \gamma$

<u>Lemma</u> (The homotopy lifting property) Given $H : (I^2, (0,0)) \to (B, b_0)$ $\exists!$ a lift $\tilde{H} : (I^2, (0,0)) \to (E, e_0)$.

done line

Thm $\pi_1(S^1,1) \cong \mathbb{Z}$

Pf $\pi_1(S^1,1) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}$

with $\phi(n)(x) = p(nx)$

and $\psi([\gamma]) = \tilde{\gamma}(1)$.

Fix the covering

$p : (\mathbb{R}, 0) \to (S^1, 1)$ $\quad p(x) = e^{2\pi i x}$

NTS 1. ψ is well-defined.

2. $\psi \circ \phi = id_{\mathbb{Z}}$ $\quad$ 3. $\phi \circ \psi = id_{\pi_1(S^1, \mathbb{Z})}$

4. ϕ is additive: $\phi(n+m) = \phi(n) * \phi(m)$ } HW!

HW9.

1. The two definitions of coverings are equivalent.
2. \phi is additive.
3. Lifting for simply connected spaces.

The path/homotopy lifting property. Given a covering $p:(E,e_0)\to(B,b_0)$ and $\begin{matrix}\gamma:(I,0)\\ H:(I^2,0)\end{matrix}\to(B,b_0)$, $\exists!$ a lift $\begin{matrix}\tilde\gamma:(I,0)\\ \tilde H:(I^2,0)\end{matrix}\to(E,e_0)$ s.t. $\begin{matrix}p\circ\tilde\gamma=\gamma\\ p\circ\tilde H=H.\end{matrix}$

Fix the covering $p:(\mathbb{R},0)\to(S^1,1)$ $p(x)=e^{2\pi i x}$

Thm $\pi_1(S^1,1)\cong\mathbb{Z}$

Pf $\pi_1(S^1,1)\underset{\psi}{\overset{\phi}{\rightleftarrows}}\mathbb{Z}$

with $\phi(n)(x)=p(nx)$

and $\psi([\gamma])=\tilde\gamma(1)$.

NTS 1. ψ is well-defined.

2. $\psi\circ\phi=id_{\mathbb{Z}}$ 3. $\phi\circ\psi=id_{\pi_1(S^1,1)}$

4. ϕ is additive: $\phi(n+m)=\phi(n)*\phi(m)$ } HW?

[V.S., L.T, compositions] [Top spaces, cont. fncts, comp]

[groups, homo, comp] [sets, fncts, comp]

Def A Category $\mathcal{C}=(\mathrm{Obj}_{\mathcal{C}},\mathrm{Mor},\circ,Id)$

s.t. Associativity, Identity

Examples The above + based spaces + homotopy classes of paths, + the game of 15 + tangles.

Def $F: \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ & $\mathcal{D}$ are categories

1. preserves compositions
2. preserves identities.

Examples: Forget, $\times 3$, $*: Vect \to Vect^{op}$, $**: Vect \circlearrowleft$

$\boxed{\pi_1}$

Thm $\nexists\, r: D^2 \to S^1$ s.t. $r|_{S^1} = Id_{S^1}$

Thm The Brouwer F.P. Theorem.