© | Dror Bar-Natan: Classes: 2022-23:

Fast Computations in Knot Theory

Tsuda University, June-July 2023

http://drorbn.net/23-FC

Tagline. A half is better than a whole!

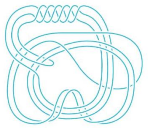
Idea. Do the computational side of Piccirillo's "The Conway Knot is Not Slice", Ann. of Math. (2) 191(2): 581-591 (March 2020), arXiv:1808.02923 (see also an article in Quanta Magazine).

Course Purpose and Content / Learning Objectives. Learn about the Jones polynomial and about Khovanov homology, and how to compute them, and how to use "tangles" to compute them even faster. Along the way learn a bit about homology theory and about category theory. Actually implement some of the algorithms learned!

Dror Bar-Natan: Classes: 2022-23: FastComputations: http://drorbn.net/23-FC/Plan.pdf

A Short Course On "**Fast Computations in Knot Theory**"

Dror Bar-Natan at Tsuda University, June 29 – July 10, 2023.



Piccirillo's Knot (Quanta Magazine)

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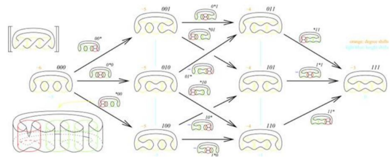
Preliminaries. Absolute confidence with linear algebra: vector spaces, linear transformations, kernels, images, Gaussian elimination. Better if you know "tensor product" and "homology" even if just barely.

Reading Preliminaries. Before the start of the course you must read the Quanta Magazine article (even without fully understanding it), and you should skim through the Piccirillo paper.

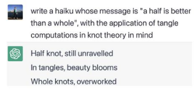
Tentative Hourly Plan.

Jones Khovanov Piccirillo

1. A quick introduction to knot theory.
2. The Jones polynomial.
3. Computing the Jones polynomial.
4. A half is better than a whole: Computing the Jones polynomial much faster.
5. Cows are better than numbers! Complexes are not so bad either.
6. Khovanov homology: The definition.
7. Homology of spaces.
8. How to prove things about complexes?
9. Khovanov homology: Invariance.
10. Khovanov homology: Computation.
11. Categories and complexes in a category.
12. Homotopy in topology and in algebra.
13. Khovanov homology for tangles.
14. Formal Gaussian elimination and delooping.
15. FastKh / a meta-half is better than a meta-whole.



Evaluation Method. Attendance (40%) and Homework (60%).



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1. Dror Bar-Natan, "On Khovanov's Categorification of the Jones Polynomial", Algebraic and Geometric Topology 2-16 (2002) 337-370 and http://www.math.toronto.edu/~drorbn/papers/Categorification/.
2. Dror Bar-Natan, "Khovanov's Homology for Tangles and Cobordisms", Geometry and Topology 9-33 (2005) 1443-1499 and http://www.math.toronto.edu/~drorbn/papers/Cobordism/.
3. Dror Bar-Natan, "Fast Khovanov Homology Computations", Journal of Knot Theory and Its Ramifications, 16-3 (2007) 243-255 and http://www.math.toronto.edu/~drorbn/papers/FastKh/.
4. Allen Hatcher, "Algebraic Topology", https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.
5. Mikhail Khovanov, "A Categorification of the Jones Polynomial", Duke Math. J. 101 (2000), no. 3, 359-426 and https://arxiv.org/abs/math/9908171.
6. Erica Klarreich, "Graduate Student Solves Decades-Old Conway Knot Problem", Quanta Magazine on May 19 2020 and https://www.quantamagazine.org/graduate-student-solves-decades-old-conway-knot-problem-20200519/.
7. Louis H. Kauffman, "On Knots", Princeton University Press 1988.
8. W. B. Raymond Lickorish, "An Introduction to Knot Theory", GTM 175, Springer 1997.
9. Lisa Piccirillo, "The Conway knot is not slice", Ann. of Math. (2) 191(2): 581-591 (March 2020) and arXiv:1808.02923.

Preliminaries. Absolute confidence with linear algebra: vector spaces, linear transformations, kernels, images, Gaussian elimination. Better if you know "tensor product" and "homology" even if just barely.

Reading Preliminaries. Before the start of the course you must read the Quanta Magazine article (even without fully understanding it), and you should skim through the Piccirillo paper.

Evaluation Method. Attendance (40%) and Homework (3 assignments, 20% each).

Day 1 - Thursday June 29, 9:30-12 and 1-2:30.

- A quick introduction to knot theory.
  See Day1Gallery.png.
- The Jones polynomial.
- Computing the Jones polynomial.

HW1 will be posted here by midnight and will be due on Monday July 3.

Day 2 - Friday June 30, 9:30-12.

- A half is better than a whole: Computing the Jones polynomial much faster.
- Cows are better than numbers! Complexes are not so bad either.

Day 3 - Monday July 3, 9:30-12 and 1-2:30.

- Khovanov homology: The definition.
- Homology of spaces.
- How to prove things about complexes?

Day 4 - Wednesday July 5, 9:30-12.

- Khovanov homology: Invariance.
- Khovanov homology: Computation.
- Categories and complexes in a category.

HW2 will be posted here by midnight and will be due on Monday July 10.

Day 5 - Friday July 7, 9:30-12 and 1-2:30.

- Homotopy in topology and in algebra.
- Khovanov homology for tangles.

Day 6 - Monday July 10, 9:30-12.

- Formal Gaussian elimination and delooping.
- FastKh / a meta-half is better than a meta-whole.

HW3 will be posted here by midnight and will be due on Thursday July 13.

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1. Dror Bar-Natan, On Khovanov's Categorification of the Jones Polynomial, Algebraic and Geometric Topology 2-16 (2002) 337-370.
2. Dror Bar-Natan, Khovanov's Homology for Tangles and Cobordisms, Geometry and Topology 9-33 (2005) 1443-1499.
3. Dror Bar-Natan, Fast Khovanov Homology Computations, Journal of Knot Theory and Its Ramifications, 16-3 (2007) 243-255.
4. Allen Hatcher, Algebraic Topology.
5. Mikhail Khovanov, A Categorification of the Jones Polynomial, Duke Math. J. 101 (2000), no. 3, 359-426.
6. Erica Klarreich, Graduate Student Solves Decades-Old Conway Knot Problem, Quanta Magazine on May 19 2020.
7. Louis H. Kauffman, "On Knots", Princeton University Press 1988.
8. W. B. Raymond Lickorish, "An Introduction to Knot Theory", GTM 175, Springer 1997.
9. Lisa Piccirillo, The Conway knot is not slice, Ann. of Math. (2) 191(2): 581-591 (March 2020).02923.

Further resources.

- Previous knot theory classes that I've given: 273a - Knot Theory as an Excuse (Harvard, 1994), 273b - Knot Theory as an Excuse (Harvard, 1995), Seminar on Knots and Lie Algebras (Jerusalem, 1997), Three Dimensional Manifolds (Jerusalem, 1998), Knot Theory (Jerusalem, 2001), Knots and Feynman Diagrams (Jerusalem, 2001), Seminar on Knot Theory (Jerusalem, 2002), 1350F - Knot Theory (Toronto, 2003), 1350F - Algebraic Knot Theory (Toronto, 2006), 1352S - Algebraic Knot Theory, (Toronto, 2007), 1350F - Algebraic Knot Theory, (Toronto, 2009), The wClips Seminar, (Toronto, 2012), (u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra), (Aarhus, 2013), 1350S - Algebraic Knot Theory, (Toronto, 2014), 1350S - Algebraic Knot Theory - Poly-Time Computations, (Toronto, 2017), 1350F - Topics in Knot Theory, (Toronto, 2020). 1350F - Topics in Knot Theory, (Toronto, 2021).
- My 23-FastComputations Pensieve Folder.

Annals of Mathematics **191** (2020), 581–591
https://doi.org/10.4007/annals.2020.191.2.5

The Conway knot is not slice

By Lisa Piccirillo

Abstract

A knot is said to be slice if it bounds a smooth properly embedded disk in B^4. We demonstrate that the Conway knot is not slice. This completes the classification of slice knots under 13 crossings and gives the first example of a non-slice knot which is both topologically slice and a positive mutant of a slice knot.

1. Introduction

The classical study of knots in S^3 is 3-dimensional; a knot is defined to be trivial if it bounds an embedded disk in S^3. Concordance, first defined by Fox in [Fox62], is a 4-dimensional extension; a knot in S^3 is trivial in concordance if it bounds an embedded disk in B^4. In four dimensions one has to take care about what sort of disks are permitted. A knot is *slice* if it bounds a smoothly embedded disk in B^4 and *topologically slice* if it bounds a locally flat disk in B^4. There are many slice knots which are not the unknot and many topologically slice knots which are not slice.

It is natural to ask how characteristics of 3-dimensional knotting interact with concordance, and questions of this sort are prevalent in the literature. Modifying a knot by positive mutation is particularly difficult to detect in concordance; we define positive mutation now.

A *Conway sphere* for an oriented knot K is an embedded S^2 in S^3 that meets the knot transversely in four points. The Conway sphere splits S^3 into two 3-balls, B_1 and B_2, and K into two tangles K_{B_1} and K_{B_2}. Any knot K^* obtained from K_{B_1} and K_{B_2} after regluing B_1 to B_2 via an involution of the Conway sphere is called a *mutant* of K. If the involution respects the orientation on K, then K^* is a *positive mutant* of K. See Figure 1 for an example.[1]

Positive mutation preserves many three dimensional invariants of a knot, including the Alexander, Jones, HOMFLY, and Kauffman polynomials [MT88],

Keywords: 4-manifolds, knot concordance, Conway mutation
AMS Classification: Primary: 57M25, 57R65.

[1]All colored figures are viewable in the online version of the article https://doi.org/10.4007/annals.2020.191.1.5.

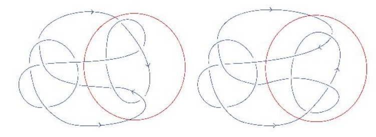

Figure 1. Positive mutation from the Conway knot (left) to the Kinoshita-Teresaka knot.

the S-equivalence class [KL01], hyperbolic volume [Rub87], and 2-fold branched cover. Other powerful invariants are conjectured to be preserved by mutation, such as Khovanov homology [BN05] and the δ-graded knot Floer groups [BL12]. Studying the sliceness of knots which arise as a positive mutant of a slice knot is even trickier: all abelian and all but the subtlest metabelian sliceness obstructions vanish for such a knot, Rasmussen's s-invariant is conjectured to vanish [BN02], and it is unknown whether any Heegaard Floer sliceness obstructions can detect such a knot.

The smallest pair of positive mutant knots, the 11 crossing Conway knot C and Kinoshita-Teresaka knot, were discovered by Conway in [Con70]; see Figure 1. These knots are also the smallest nontrivial knots with Alexander polynomial 1, hence all of their abelian and metabelian sliceness obstructions vanish, and by Freedman [Fre84] both knots are topologically slice. The Conway and Kinoshita-Teresaka knots were first distinguished in isotopy by Riley in [Ril71] via careful study of their groups, and later their Seifert genera were distinguished by Gabai [Gab86]. One can readily show that the Kinoshita-Teresaka knot is slice. Despite the wealth of sliceness obstructions constructed in the past 20 years which are not known to be mutation invariant and do not necessarily vanish for Alexander polynomial 1 knots, it has remained open whether the Conway knot is slice.

In 2001 Kirk and Livingston gave the first examples of non-slice knots which are positive mutants of slice knots [KL01], and other examples have appeared since [KL05], [HKL10], [Mil17]. All of these works rely on careful analysis of metabelian sliceness obstructions. Since metabelian obstructions obstruct topological sliceness, these techniques cannot detect any topologically slice knot which is a mutant of a slice knot, and they are especially poorly suited to an Alexander polynomial 1 knot such as the Conway knot.

Theorem 1.1. *The Conway knot is not slice.*

This completes the classification of slice knots of under 13 crossings [CL05] and gives the first example of a non-slice knot which is both topologically slice

and a positive mutant of a slice knot.

Since there are no sliceness obstructions known to be well suited to detecting the Conway knot, we abandon the Conway knot. We will instead construct a knot K' such that the Conway knot is slice if and only if K' is slice, and we concern ourselves with detecting the sliceness of this K'. (Experts will note that our K' is not concordant to the Conway knot.) To construct such a K' we will be interested in the following manifold.

Definition 1.2. A knot trace $X(K)$ is a four manifold obtained by attaching a 0-framed 2-handle to B^4 with attaching sphere K.

We will use $\cong$ to denote diffeomorphisms of manifolds. The following observation is folklore; for an early, use see [KM78].

Lemma 1.3. *K is slice if and only if $X(K)$ smoothly embeds in S^4.*

Proof. For the "only if" direction. Consider S^4 and a smooth S^3 therein which decomposes S^4 into the union of two 4-balls B_1 and B_2. Consider K sitting in this S^3. Since K is slice, we can find a smoothly embedded disk D_K which K bounds in B_1. Observe now that $B_2 \cup \overline{\nu(D_K)} \cong X(K)$ is smoothly embedded in S^4.

For the "if" direction. Let $F: S^2 \to X(K)$ be a piecewise linear embedding such that the image of F consists of the union of the cone on K with the core of the 2-handle. Notice that F is smooth away from the cone point p. Let $i: X(K) \to S^4$ be a smooth embedding. Then $(i \circ F)$ is a piecewise linear embedding of S^2 in S^4, which is smooth away from $i(p)$. Note that $W := S^4 \smallsetminus \nu(i(p)) \cong B^4$ and that the restriction of $(i \circ F)$ to the complement of a small neighborhood of $F^{-1}(p)$ in S^2 is a smooth embedding of D^2 in $W \cong B^4$. Further, if we choose this neighborhood to be the inverse image of a sufficiently small neighborhood of $i(p)$, we have that $(i \circ F)(D^2 \smallsetminus \nu(F^{-1}(p)))$ intersects ∂W in the knot K. □

Then for any knots K and K' with $X(K) \cong X(K')$, K is slice if and only if K' is slice. There exists a small body of literature on producing pairs of knots K and K' with $X(K) \cong X(K')$. We will rely here on the dualizable patterns construction which was pioneered by Akbulut in [Akb77], developed by Lickorish [Lic79] and Gompf-Miyazaki [GM95], and recently re-interpreted by the author [Pic19]. Unknotting number one knots fit into this construction (a related statement appears in [AJOT13]); using this we prove the following.

Proposition 1.4. *The knot K' in Figure 2 has $X(Conway) \cong X(K')$.*

Thus it suffices to prove that K' is not slice. The advantage of this perspective is that we do not have any reason to expect K' to be a mutant of a

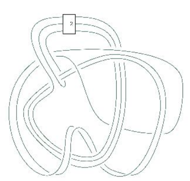

Figure 2. The knot K' shares a trace with the Conway knot.

slice knot, so we hope that not all slice obstructions for K' will vanish. This hope is complicated by the fact that K' has Alexander polynomial 1, thus all its abelian and metabelian sliceness invariants vanish. In general it is difficult to distinguish knots with diffeomorphic traces (see [Kir78, Prob. 1.21]), however, recent work of the author [Pic19] and Miller and the author [MP18] demonstrate that Rasmussen's s-invariant and the Heegaard Floer correction terms of the double branched covers can be used to distinguish such knots. Using the s-invariant, we show

Theorem 1.5. *K' is not slice.*

The s-invariant shares many formal properties with Ozsváth-Szabó's τ-invariant. However, it follows from recent work of Hayden, Mark, and the author [HMP19] that for any K' with $X(Conway) \cong X(K')$, $\epsilon(K') = 0$ and hence $\tau(K') = 0$.

In Section 2 we construct K' and prove Proposition 1.4. In Section 3 we compute $s(K')$ and prove Theorem 1.5. We will assume familiarity with handle calculus; for details see [GS99].

Acknowledgments. The author was reminded of this problem during Shelly Harvey's talk at Bob Gompf's birthday conference. The author is in frequent conversation with Allison N. Miller, and those insightful conversations inform this work. The author is deeply indebted to her advisor John Luecke, whose constant encouragement, context, and insights are indispensable to her work.

2. Constructing K′ which shares a trace with the Conway knot

We begin by recalling the dualizable patterns construction, as presented in [Pic19]. Let L be a three component link with (blue, green, and red) components B, G, and R such that the following hold: the sublink $B \cup R$ is isotopic in S^3 to the link $B \cup \mu_B$, where μ_B denotes a meridian of B, the sublink $G \cup R$

is isotopic to the link $G \cup \mu_G$, and $\mathrm{lk}(B, G) = 0$. We will call such a link a *dualizable link*. From a dualizable link we can define an associated four manifold X by thinking of R as a 1-handle, in dotted circle notation, and B and G as attaching spheres of 0-framed 2-handles. A dualizable link L also defines a pair of associated knots K and K'; we will give the association in the proof of Theorem 2.1.

Theorem 2.1 ([Pic19]). *If L is a dualizable link with X the associated 4-manifold and K and K' the associated knots, then $X \cong X(K) \cong X(K')$.*

Proof. Isotope L to a diagram in which R has no self crossings (hence such that R bounds a disk D_R in the diagram) and in which $B \cap D_R$ is a single point. Slide G over B as needed to remove the intersections of G with D_R. After the slides we can cancel the 2-handle with attaching circle B with the 1-handle, and we are left with a handle description for a 0-framed knot trace; this knot is K'.

To construct K and see $X \cong X(K)$, perform the above again with the roles of B and G reversed. □

Proposition 2.2. *For any unknotting number 1 knot C, there exists a dualizable link L such that 4-manifold X associated to L is diffeomorphic to $X(C)$.*

Proof. We will prove the claim for C admitting a positive unknotting crossing c; the proof for c negative is similar. Define knots R and G in $S^3 \smallsetminus \nu(C)$ as in the left frame of Figure 3, where R is a blackboard parallel of D outside of the diagram. Define X to be the four manifold obtained by thinking of R as a 1-handle in dotted circle notation, and attaching 0-framed 2-handles along K and G. Since G and R are a canceling 1-2 pair, we see that $X \cong X(K)$. To finish the proof we will perform handle slides to get a dualizable link L such that X is the associated 4-manifold.

To this end, slide C over R as indicated in Figure 3 to get a handle description for X as in the center frame. Observe that the attaching sphere for

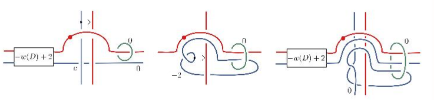

Figure 3. Constructing a dualizable link L associated to an unknotting number 1 knot C. Here D denotes a (blue) diagram of C with a positive unknotting crossing c, and $w(D)$ denotes the writhe of D.

the curve indicated in blue in the center frame is isotopic to a meridian of R. As such, performing the indicated slide to get the right frame will yield a link L with 0-framed blue attaching sphere B which can be isotoped so that $B \cup R$ is isotopic to $B \cup \mu_B$, and one observes that $\mathrm{lk}(B, G) = 0$. □

Thus for any unknotting number one knot C, one can produce a link L as in Theorem 2.1 and use L to produce a knot K' with $X(K') \cong X(C)$. We remark that the unknotting number one knot C is in fact isotopic to the knot K produced from L as in the proof of Theorem 2.1, though we will not rely on that here.

Proof of Proposition 1.4. We now produce such a K' for the Conway knot. We proceed as in the proofs of Proposition 2.2 and Theorem 2.1; in order to produce a diagram of K' with small crossing number, we will perform additional isotopies throughout. See Figure 4. □

3. Showing K′ is not slice

In [Kho00] Khovanov introduced a link invariant $Kh^{i,j}(L)$ which is the (co)homology of a finitely generated bigraded chain complex $(C^{i,j}(D_L), d)$. In our notation, D_L denotes a diagram of L and i is referred to as the homological grading and j, the quantum grading. Later Lee [Lee02] introduced a modification of the Khovanov differential: she considered instead a graded filtered complex $(C^{i,j}(D_L), d')$, such that d' raises homological grading by 1 and for any homogeneous $v \in C^{i,j}(D_L)$, the quantum grading of every monomial in $d'(v)$ is greater than or equal to the quantum grading of v. As a consequence of her construction, there exists a spectral sequence with $(E_1^{i,j}(D_L), d_1) = (C^{i,j}(D_L), d)$ and $E_2^{i,j} = Kh^{i,j}(L)$ which converges to the homology of the Lee complex for L. We will denote this homology group $KhL^{i,j}(L)$. It will be relevant for us that the differentials d_n of the spectral sequence have bidegree $(1, 4(n-1))$ (see [Ras10]). Lee proves that for any knot K, $KhL(K) = \mathbb{Q} \oplus \mathbb{Q}$ where both generators are located in grading $i = 0$. Rasmussen used this to define an integer valued knot invariant $s(K)$ as follows.

Theorem 3.1 ([Ras10]). *For any knot K, the generators of Lee homology are located in gradings $(i, j) = (0, s(K) \pm 1)$. If K is slice, then $s(K) = 0$.*

Proof of Theorem 1.5. Let K' be the knot from Proposition 1.4; to show K' is not slice we will calculate $s(K')$. To begin, we compute the Khovanov homology of K', using Bar-Natan's Fast-Kh routines available at [KAT]. These routines produce the polynomial $Kh(K)(t, q) := \Sigma_{i,j} t^i q^j \mathrm{rank}(Kh^{i,j}(K) \otimes \mathbb{Q})$. We plot the values $\mathrm{rank}(Kh^{i,j}(K') \otimes \mathbb{Q})$ in Table 1.

Since the Lee homology is supported in grading $i = 0$, we see that $s(K') \in \{0, 2\}$. To demonstrate that in fact $s(K') = 2$, we will use the fact that all

higher differentials in the spectral sequence to the Lee homology have bidegree $(1, 4(n-1))$. Consider a generator x of $Kh^{0,3}(K')$. If x were to die on the n^{th} page of the spectral sequence $(n \geq 2)$, we would need to have that either $d_n(x) \neq 0$ or there exists a y with $d_n(y) = x$. Since $Kh^{i,j}(K')$ has no generators in gradings $\{1, 4(n-1)+3\}$ or $\{-1, -4(n-1)+3\}$ for any $n \geq 2$, neither of these can happen. As such, x survives to the E^∞ page, and $s(K') = 2$. □

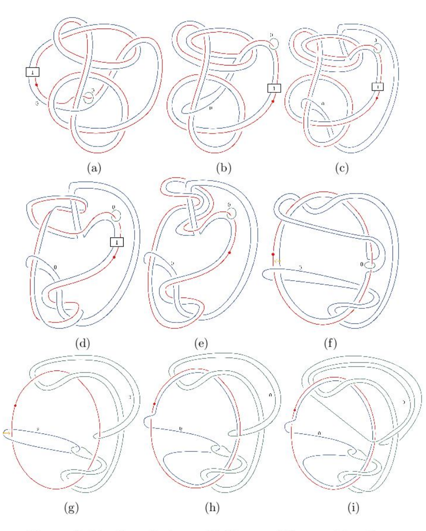

Figure 4. Handle calculus exhibiting a diffeomorphism from $X(C)$ to $X(K')$ where K' is the knot defined in Figure 2. Handle slides are denoted with arrows, the transition from (L) to (M) includes canceling a 1-2 pair, and all other changes are isotopies.

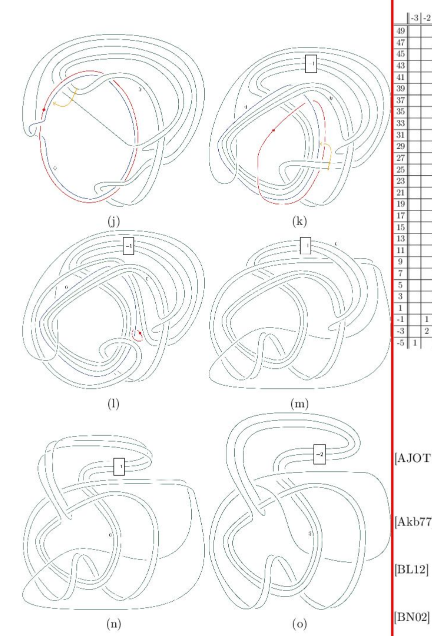

Figure 4. continued

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Table 1

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KNOT THEORY

Graduate Student Solves Decades-Old Conway Knot Problem

By ERICA KLARREICH

May 19, 2020

It took Lisa Piccirillo less than a week to answer a long-standing question about a strange knot discovered over half a century ago by the legendary John Conway.

Lisa Piccirillo's solution to the Conway knot problem helped her land a tenure-track position at the Massachusetts Institute of Technology.

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Ian MacLellan for Quanta Magazine

In the summer of 2018, at a conference on low-dimensional topology and geometry, Lisa Piccirillo heard about a nice little math problem. It seemed like a good testing ground for some techniques she had been developing as a graduate student at the University of Texas, Austin.

"I didn't allow myself to work on it during the day," she said, "because I didn't consider it to be real math. I thought it was, like, my homework."

The question asked whether the Conway knot — a snarl discovered more than half a century ago by the legendary mathematician John Horton Conway — is a slice of a higher-dimensional knot. "Sliceness" is one of the first natural questions knot theorists ask about knots in higher-dimensional spaces, and mathematicians had been able to answer it for all of the thousands of knots with 12 or fewer crossings — except one. The Conway knot, which has 11 crossings, had thumbed its nose at mathematicians for decades.

Before the week was out, Piccirillo had an answer: The Conway knot is not "slice." A few days later, she met with Cameron Gordon, a professor at UT Austin, and casually mentioned her solution.

"I said, 'What?? That's going to the *Annals* right now!'" Gordon said, referring to *Annals of Mathematics*, one of the discipline's top journals.

"He started yelling, 'Why aren't you more excited?'" said Piccirillo, now a postdoctoral fellow at Brandeis University. "He sort of freaked out."

"I don't think she'd recognized what an old and famous problem this was," Gordon said.

Piccirillo's proof appeared in *Annals of Mathematics* in February. The paper, combined with her other work, has secured her a tenure-track job offer from the Massachusetts Institute of Technology that will begin on July 1, only 14 months after she finished her doctorate.

The question of the Conway knot's sliceness was famous not just because of how long it had gone unsolved. Slice knots give mathematicians a way to probe the strange nature of four-dimensional space, in which two-dimensional spheres can be knotted, sometimes in such crumpled ways that they can't be smoothed out. Sliceness is "connected to some of the deepest questions in four-dimensional topology right now," said Charles Livingston, an emeritus professor at Indiana University.

"This question, whether the Conway knot is slice, had been kind of a touchstone for a lot of the modern developments around the general area of knot theory," said Joshua Greene of Boston College, who supervised Piccirillo's senior thesis when she was an undergraduate there. "It was really gratifying to see somebody I'd known for so long suddenly pull the sword from the stone."

Magic Spheres

While most of us think of a knot as existing in a piece of string with two ends, mathematicians think of the two ends as joined, so the knot can't unravel. Over the past century, these knotted loops have helped illuminate subjects from quantum physics to the structure of DNA, as well as the topology of three-dimensional space.

But our world is four-dimensional if we include time as a dimension, so it is natural to ask if there is a corresponding theory of knots in 4D space. This isn't just a matter of taking all the knots we have in 3D space and plunking them down in 4D space: With four dimensions to move around in, any knotted loop can be unraveled if strands are moved over each other in the fourth dimension.

To make a knotted object in four-dimensional space, you need a two-dimensional sphere, not a one-dimensional loop. Just as three dimensions provide enough room to build knotted loops but not enough room for them to unravel, four dimensions provide such an environment for knotted spheres, which mathematicians first constructed in the 1920s.

It's hard to visualize a knotted sphere in 4D space, but it helps to first think about an ordinary sphere in 3D space. If you slice through it, you'll see an unknotted loop. But when you slice through a knotted sphere in 4D space, you might see a knotted loop instead (or possibly an unknotted loop or a link of several loops, depending on where you slice). Any knot you can make by slicing a knotted sphere is said to be "slice." Some knots are not slice — for instance, the three-crossing knot known as the trefoil.

Slice knots "provide a bridge between the three-dimensional and four-dimensional stories of knot theory," Greene said.

But there's a wrinkle that lends richness and peculiarity to the four-dimensional story: In 4D topology, there are two different versions of what it means to be slice. In a series of revolutionary developments in the early 1980s (which earned both Michael Freedman and Simon Donaldson Fields Medals), mathematicians discovered that 4D space doesn't just contain the smooth spheres we intuitively visualize — it also contains spheres so pervasively crumpled that they could never be ironed smooth. The question of which knots are slice depends on whether you choose to include these crumpled spheres.

"These are very, very strange objects, that sort of exist by magic," said Shelly Harvey of Rice University. (It was at Harvey's talk in 2018 that Piccirillo first learned about the Conway knot problem.)

These strange spheres are not a bug of four-dimensional topology, but a feature. Knots that are "topologically slice" but not "smoothly slice" — meaning they are a slice of some crumpled sphere, but no smooth one — allow mathematicians to build so-called "exotic" versions of ordinary four-dimensional space. These copies of four-dimensional space look the same as normal space from a topological viewpoint but are irretrievably crumpled. The existence of these exotic spaces sets dimension four apart from all other dimensions.

The question of sliceness is "the lowest-dimensional probe" of these exotic four-dimensional spaces, Greene said.

Over the years, mathematicians discovered an assortment of knots that were topologically but not smoothly slice. Among knots with 12 or fewer crossings, however, there didn't seem to be any — except possibly the Conway knot. Mathematicians could figure out the slice status of all other knots with 12 or fewer crossings, but the Conway knot eluded them.

Conway, who died of COVID-19 last month, was famous for making influential contributions to one area of mathematics after another. He first became interested in knots as a teenager in the 1950s and came up with a simple way to list essentially all the knots up to 11 crossings (previous complete lists had gone up to only 10 crossings).

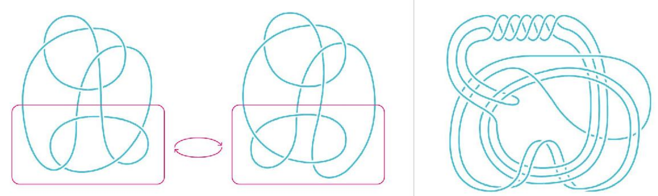
On the list was one knot that stood out. "Conway, I think, realized that there was something quite special about it," Greene said.

The Conway knot, as it came to be known, is topologically slice — mathematicians realized this amid the revolutionary discoveries of the 1980s. But they couldn't figure out whether it was smoothly slice. They suspected that it was not, because it seemed to lack a feature called "ribbonness" that smoothly slice knots typically have. But it also had a feature that made it immune to every attempt to show it was not smoothly slice.

Namely, the Conway knot has a sort of sibling — what's known as a mutant. If you draw the Conway knot on paper, cut out a certain portion of the paper, flip the fragment over and then rejoin its loose ends, you get another knot known as the Kinoshita-Terasaka knot.

A Knotty Problem

When researchers wanted to prove that the Conway knot is not "slice," they were hamstrung by its close resemblance to another knot called the Kinoshita-Terasaka knot, which is slice. Then Lisa Piccirillo figured out how to make the Conway knot a new, more complicated companion knot.



The Conway knot and the Kinoshita-Terasaka knot are mutants, meaning you can transform one into the other by flipping the part of the knot within the red box.

Piccirillo devised a knot with the same "trace" as the Conway knot, then used this new knot to ascertain the Conway knot's slice status.

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5W Infographics for Quanta Magazine

The trouble is, this new knot happens to be smoothly slice. And because the Conway knot is so closely related to a smoothly slice knot, it manages to hoodwink all the tools (called invariants) that mathematicians use to detect non-slice knots.

"Whenever a new invariant comes along, we try to test it against the Conway knot," Greene said. "It's just this one stubborn example that, it seems, no matter what invariant you come up with, it won't tell you whether or not the thing is slice."

The Conway knot "sits at the intersection of the blind spots" of these different tools, Piccirillo said.

One mathematician, Mark Hughes of Brigham Young University, created a neural network that uses knot invariants and other information to make predictions about features such as sliceness. For most knots, the network makes clear predictions. But its guess about whether the Conway knot is smoothly slice? Fifty-fifty.

"Over time it stood out as the knot that we couldn't handle," Livingston said.

Clever Twists

Piccirillo enjoys the visual intuition that knot theory entails, but she doesn't think of herself primarily as a knot theorist. "It's really [three- and four-dimensional shapes] that are exciting for me, but the study of these things is deeply linked with knot theory, so I do a bit of that too," she wrote in an email.

When she first started studying mathematics in college, she didn't stand out as a "standard golden child math prodigy," said Elisenda Grigsby, one of Piccirillo's professors at Boston College. Rather, it was Piccirillo's creativity that caught Grigsby's eye. "She believed very much in her own point of view, and always has."

Piccirillo encountered the question about the Conway knot at a time when she was pondering another way two knots can be related besides mutation. Every knot has an associated four-dimensional shape called its trace, which is made by placing the knot on the boundary of a 4D ball and sewing a sort of cap onto the ball along the knot. A knot's trace "encodes that knot in a very strong way," Gordon said.

One of Piccirillo's former professors cited creativity as one of her core strengths as a mathematician.

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Ian MacLellan for Quanta Magazine

Different knots can have the same four-dimensional trace, and mathematicians already knew that these trace siblings, so to speak, always have the same slice status — either they're both slice, or they're both not slice. But Piccirillo and Allison Miller, now a postdoctoral fellow at Rice, had shown that these trace siblings don't necessarily look the same to all the knot invariants used to study sliceness.

That pointed Piccirillo toward a strategy for proving that the Conway knot is not slice: If she could construct a trace sibling for the Conway knot, maybe it would cooperate with one of the slice invariants better than the Conway knot does.

Constructing trace siblings is a tricky business, but Piccirillo was an expert. "That's just, like, a trade I'm in," she said. "So I just went home and did it."

Through a combination of clever twists, Piccirillo managed to construct a complicated knot that has the same trace as the Conway knot. For that knot, a tool called Rasmussen's s-invariant shows it is not smoothly slice — so the Conway knot can't be either.

"It's a really beautiful proof," Gordon said. There was no reason to expect that the knot Piccirillo constructed would yield to Rasmussen's s-invariant, he said. "But it worked … kind of amazingly."

Piccirillo's proof "fits into the mold of short, surprising proofs of elusive results that researchers in the area are able to quickly absorb, admire and seek to generalize — not to mention wonder how it took so long to come up with," Greene wrote in an email.

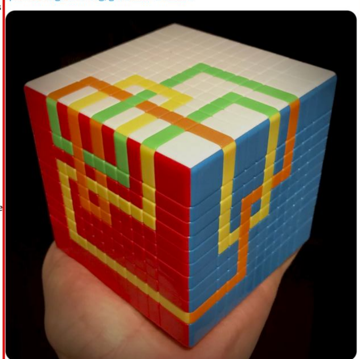
Knot traces are a classical tool that has been around for decades, but one that Piccirillo understood more deeply than anyone else, Greene said. Her work has shown topologists that knot traces are underappreciated, he said. "She's picked up some tools that maybe had a bit of dust on them," he said. "Others are following suit now."

Correction: *May 21, 2020*
The illustration depicting Piccirillo's knot has been updated to fix one of the crossings.

This article was reprinted on Wired.com and summarized on Washingtonpost.com.

Cubes_Art
@cubes_art

Congrats to Lisa Piccirillo for determining the Conway knot is *not* slice!
Nice writeup in @QuantaMagazine
Here's the Conway knot on an 11x11 Rubik's cube to celebrate. 👍👍
quantamagazine.org/graduate-stude...



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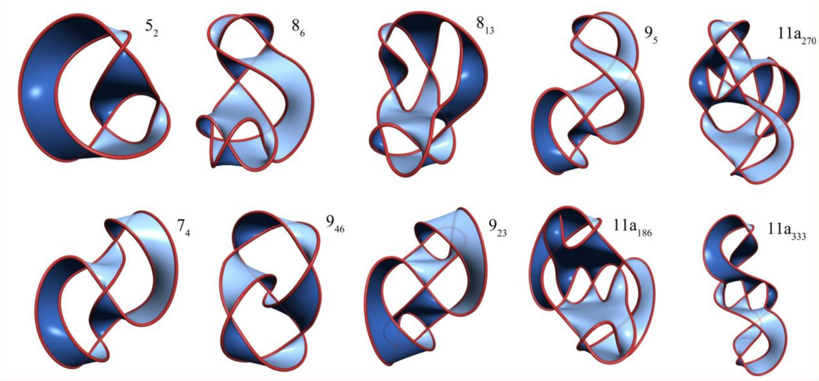
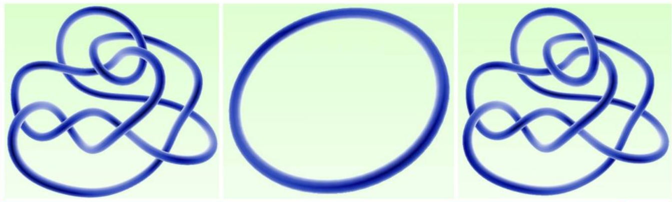
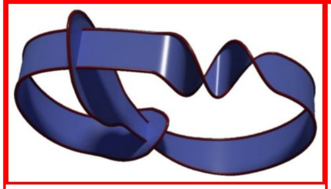
Some Seifert surfaces from Seifert Surfaces with Minimal Genus:

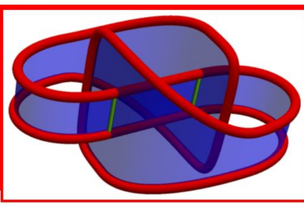
van Garderen and van Wijk



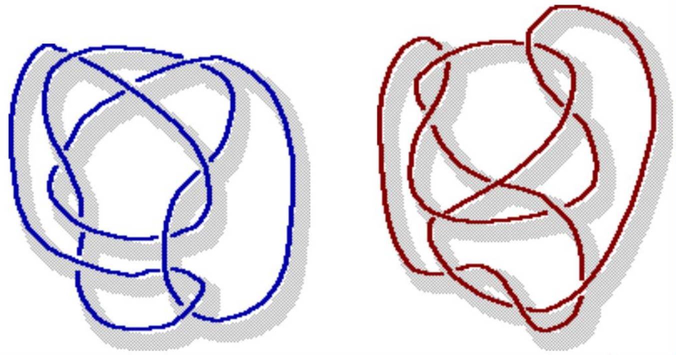
Which two are the same? (rendered using Rob Scharein's KnotPlot)







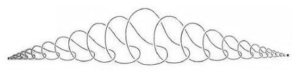
The Perko Pair: (are these the same?) (taken from http://www.math.cuhk.edu.hk/publect/lecture4/perko.html)



Dror Bar-Natan: Classes: 2001-02: Algebraic Topology: screen version print version

Topological Pathologies in R^3

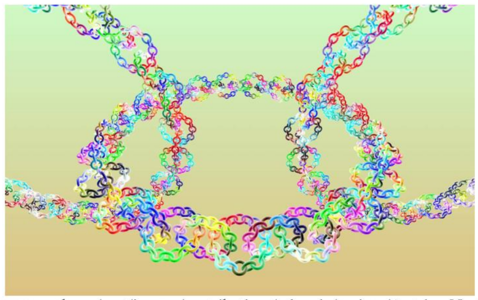
An embedding of an interval in R^3 whose complement is not simply connected:



See Hocking and Young's *Topology* pp. 176-177.

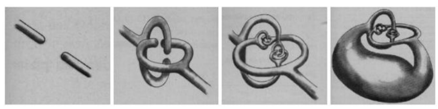
See http://www.math.ohio-state.edu/~fiedorow/math655/Jordan.html.

Antoine's necklace - an embedding of a Cantor set in R^3 whose complement is not simply connected:

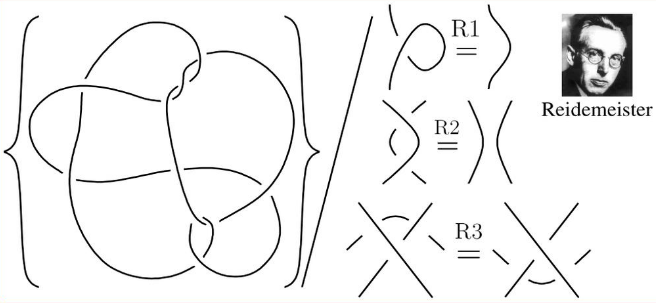


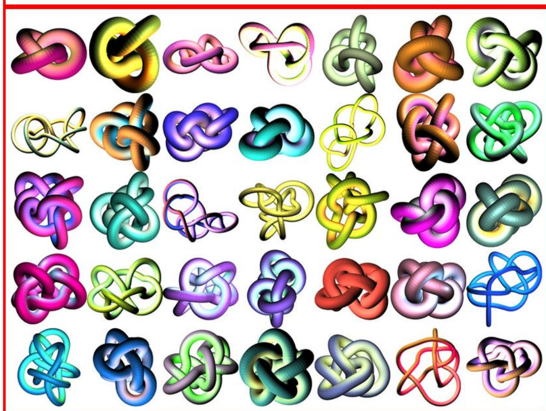
See http://www.cs.ubc.ca/nest/imager/contributions/scharein/various/AntoinesNecklace.html.

The Alexander horned sphere - a continuous embedding of a ball in R^3 whose complement is not simply connected:



See http://users.math.uni-potsdam.de/~oeitner/EIGENES/RAEUME/hornsph.htm.





6 LISA PICCIRILLO

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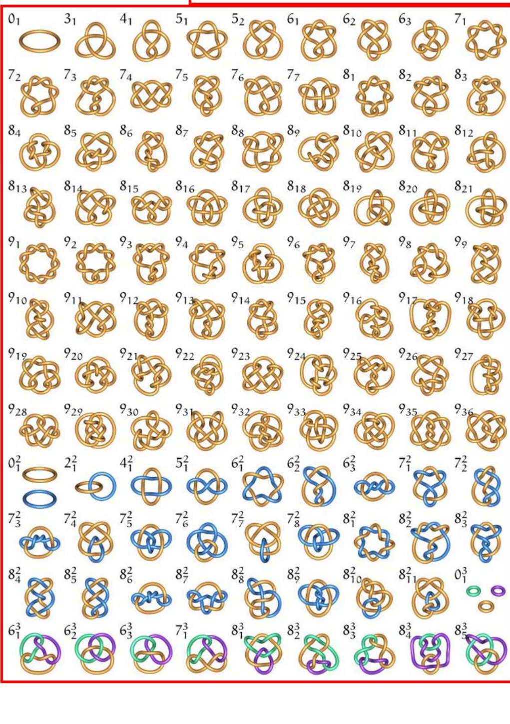
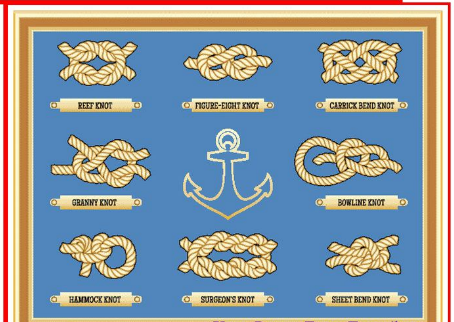
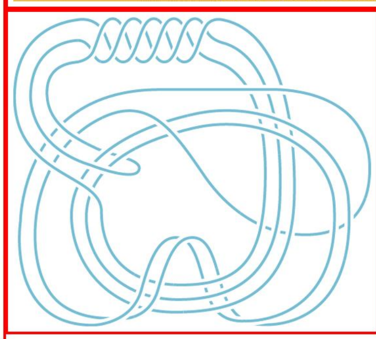
TABLE 1





Knot Poster From Freepik.com



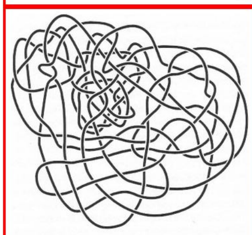
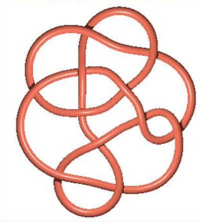


Figure 3.5. Wolfgang Haken's "Gordian knot."

The Conway Knot by Saung Tadashi

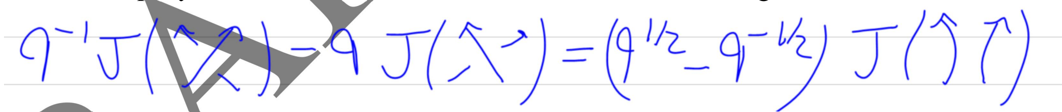


Homework Assignment 1

Solve and submit the following problems.

Problem 1.

1. Prove that the Jones polynomial satisfies the "skein relation" in the figure below.



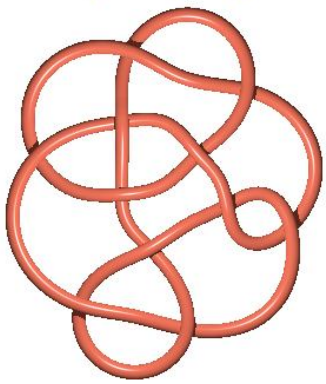
2. Show that this skein relation, along with the value of J on the unknot, determine J. Phrase your answer as "here is an algorithm to compute J on any knot/link, using only the skein relation and the value on the unknot".

Problem 2. Rather than fixing the Kauffman bracket by using a writhe counter-term, it is tempting to evaluate it at $A = e^{\pi i/3}$, where invariance under R1 holds with no need for a correction. Unfortunately, at $A = e^{\pi i/3}$ the Kauffman bracket of any knot is equal to 1. Prove this.

Problem 3. Prove that the PD notation of a knot diagram determines it as a diagram in S^2.

Problem 4. Use the programs we wrote in class to compute the Jones polynomial of the Conway Knot:

The Conway Knot by Saung Tadashi



Due date. This assignment is due on Monday July 3 at 11:59pm.