Pensieve header: The WG Algebra: testing, knots, optimization.

Loading Knot Data

```
In[•]:= Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= AllKnots[{3, 5}]
Out[•]= {Knot[3, 1], Knot[4, 1], Knot[5, 1], Knot[5, 2]}
```

```
In[•]:= PD /@ AllKnots[{3, 5}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[•]= {PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]],
 PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]],
 PD[X[1, 6, 2, 7], X[3, 8, 4, 9], X[5, 10, 6, 1], X[7, 2, 8, 3], X[9, 4, 10, 5]],
 PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1], X[9, 6, 10, 7], X[7, 2, 8, 3]]}
```

Defining a Group

```
In[•]:= DeclareGroup[S_k_] := Module[{α, β, e, γs},
   Clear[G, n, g, ι, m, inv];
   G = PermutationCycles /@ (Permutations@Range@k);
   n = Length[G];
   Do[g[α] = e = G[[α]]; ι[e] = α, {α, n}];
   m[] = ι[Cycles[{}]];
   Do[m[α, β] = ι[g[α] ~PermutationProduct~ g[β]], {α, n}, {β, n}];
   m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
   Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
  ]
```

```
In[•]:= g[3]
Out[•]= g[3]
```

```
In[•]:= k = 3; Range@k
Out[•]= {1, 2, 3}
```

```
In[•]:= Permutations@Range@k
Out[•]= {{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}}
```

```
In[•]:= Permutations[{j, l, a}]
Out[•]= {{j, l, a}, {j, a, l}, {l, j, a}, {l, a, j}, {a, j, l}, {a, l, j}}
```

```
In[•]:= PermutationCycles /@ (Permutations@Range@k)

Out[•]= {Cycles[{}], Cycles[{{2, 3}}], Cycles[{{1, 2}}],
 Cycles[{{1, 2, 3}}], Cycles[{{1, 3, 2}}], Cycles[{{1, 3}}]}

In[•]:= PermutationCycles /@ Permutations[{j, l, a}]

PermutationCycles: {j, l, a} is not a valid permutation.

PermutationCycles: {j, a, l} is not a valid permutation.

PermutationCycles: {l, j, a} is not a valid permutation.

General: Further output of PermutationCycles::perm will be suppressed during this calculation.

Out[•]= {PermutationCycles[{j, l, a}], PermutationCycles[{j, a, l}], PermutationCycles[{l, j, a}],
 PermutationCycles[{l, a, j}], PermutationCycles[{a, j, l}], PermutationCycles[{a, l, j}]}

In[•]:= DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G⟦α⟧; ι[e] = α, {α, n} ];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α]~PermutationProduct~g[β]], {α, n}, {β, n} ];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]

In[•]:= G = PermutationCycles /@ (Permutations@Range@k)

Out[•]= {Cycles[{}], Cycles[{{2, 3}}], Cycles[{{1, 2}}],
 Cycles[{{1, 2, 3}}], Cycles[{{1, 3, 2}}], Cycles[{{1, 3}}]}

In[•]:= n = Length[G]

Out[•]= 6

In[•]:= ? ι

Out[•]=
Symbol
Global`ι
Full Name  Global`ι

In[•]:= Do[g[α] = e = G⟦α⟧; ι[e] = α, {α, n} ]
```

In[∘]:= ? ι

Out[∘]=

Symbol

Global`ι

Definitions

```
ι[Cycles[{}]] = 1

ι[Cycles[{{2, 3}}]] = 2

ι[Cycles[{{1, 2}}]] = 3

ι[Cycles[{{1, 2, 3}}]] = 4

ι[Cycles[{{1, 3, 2}}]] = 5

ι[Cycles[{{1, 3}}]] = 6
```

Full Name Global`ι

In[∘]:=
```
DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G[[α]]; ι[e] = α, {α, n} ];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α]~PermutationProduct~g[β]], {α, n}, {β, n} ];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]
```

In[∘]:= ? m

Out[∘]=

Symbol

Global`m

Full Name Global`m

In[∘]:=
```
Do[m[α, β] = ι[g[α]~PermutationProduct~g[β]], {α, n}, {β, n} ]
```

In[∘]:= ? m

Symbol

Global`m

Definitions

```
m[1, 1] = 1
m[4, 4] = 5
m[3, 4] = 6
m[3, 3] = 1
m[2, 6] = 5
m[5, 1] = 5
m[5, 6] = 2
m[1, 5] = 5
m[5, 5] = 4
m[2, 3] = 4
m[5, 3] = 6
m[6, 4] = 2
m[4, 1] = 4
m[2, 5] = 6
m[6, 1] = 6
m[5, 2] = 3
m[6, 3] = 5
m[6, 6] = 1
m[3, 1] = 3
m[1, 4] = 4
m[3, 6] = 4
m[3, 2] = 5
m[2, 1] = 2
m[2, 2] = 1
m[6, 2] = 4
m[5, 4] = 1
```

Out[∘]=

```
m[4, 2] = 6

m[1, 2] = 2

m[4, 5] = 1

m[3, 5] = 2

m[1, 3] = 3

m[4, 6] = 3

m[2, 4] = 3

m[1, 6] = 6

m[4, 3] = 2

m[6, 5] = 3

Full Name  Global`m
```

```
In[]:= DeclareGroup[S_k_] := Module[{α, β, e, γs},
         Clear[G, n, g, ι, m, inv];
         G = PermutationCycles /@ (Permutations@Range@k);
         n = Length[G];
         Do[g[α] = e = G[[α]]; ι[e] = α, {α, n} ];
         m[] = ι[Cycles[{}]];
         Do[m[α, β] = ι[g[α] ~PermutationProduct~ g[β]], {α, n}, {β, n} ];
         m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
         Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
        ]

In[]:= m[inv[3], 2, 3]

Out[]= m[inv[3], 2, 3]

In[]:= DeclareGroup[S_3]

In[]:= m[inv[3], 2, 3]

Out[]= 6

In[]:= G

Out[]= {Cycles[{}], Cycles[{{2, 3}}], Cycles[{{1, 2}}],
        Cycles[{{1, 2, 3}}], Cycles[{{1, 3, 2}}], Cycles[{{1, 3}}]}
```

In[∘]:=
```
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 4 & 6 & 2 & 5 & 1 & 3 \\ 5 & 3 & 6 & 1 & 4 & 2 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

Defining WG

In[∘]:=
```
Basis[] = {1};
Basis[i_, is___] := Flatten@Table[W_i[α, β] Basis[is], {α, n}, {β, n}]
```

In[∘]:=
```
Basis[]
```

Out[∘]= {1}

In[∘]:=
```
Basis[j]
```

Out[∘]=
```
{W_j[1, 1], W_j[1, 2], W_j[1, 3], W_j[1, 4], W_j[1, 5], W_j[1, 6], W_j[2, 1], W_j[2, 2], W_j[2, 3],
 W_j[2, 4], W_j[2, 5], W_j[2, 6], W_j[3, 1], W_j[3, 2], W_j[3, 3], W_j[3, 4], W_j[3, 5], W_j[3, 6],
 W_j[4, 1], W_j[4, 2], W_j[4, 3], W_j[4, 4], W_j[4, 5], W_j[4, 6], W_j[5, 1], W_j[5, 2], W_j[5, 3],
 W_j[5, 4], W_j[5, 5], W_j[5, 6], W_j[6, 1], W_j[6, 2], W_j[6, 3], W_j[6, 4], W_j[6, 5], W_j[6, 6]}
```

In[∘]:=
```
Basis[1, 2] // Short
```

Out[∘]//Short= { ≪1≫ }

In[∘]:=
```
m_{i_,j_→k_}[ℰ_] :=
  Expand[ℰ] /. W_i[α_, β_] W_j[γ_, δ_] ⧴ If[m[α, β] == m[β, γ], W_k[α, m[β, δ]], 0];
η_{i_}[ℰ_] := Expand[ℰ Sum[W_i[α, m[]], {α, n}]];
```

In[∘]:=
```
Δ_{i_→j_,k_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ Sum[W_j[γ, β] W_k[m[α, inv[γ]], β], {γ, n}]];
ε_{i_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ If[α == m[], 1, 0]];
```

In[∘]:=
```
S_{i_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ W_i[m[inv[β], inv[α], β], inv[β]]];
```

In[∘]:=
```
R_{i_,j_} := Sum[W_i[α, m[]] W_j[β, α], {α, n}, {β, n}];
R̄_{i_,j_} := Sum[W_i[α, m[]] W_j[β, inv@α], {α, n}, {β, n}];
```

Testing the Axioms of an IHOP+R

m is associative:

In[∘]:= `b = Basis[1, 2, 3]; (b // m_{1,2→1} // m_{1,3→1}) == (b // m_{2,3→2} // m_{1,2→1})`

Out[∘]= True

In[∘]:= `b = Basis[1, 2, 3]`

Out[∘]= $\{W_1[1,1]\,W_2[1,1]\,W_3[1,1],\ W_1[1,1]\,W_2[1,1]\,W_3[1,2],$ … 46653 … $, W_1[6,6]\,W_2[6,6]\,W_3[6,6]\}$

large output | show less | show more | show all | set size limit...

In[∘]:= `b = RandomChoice[Basis[1, 2, 3]]`

Out[∘]= $W_1[2,3]\,W_2[6,2]\,W_3[3,1]$

In[∘]:= `(b // m_{1,2→1} // m_{1,3→1})`

Out[∘]= $W_1[2,5]$

In[∘]:= `(b // m_{2,3→2} // m_{1,2→1})`

Out[∘]= $W_1[2,5]$

In[∘]:= `b = Basis[1, 2, 3]`

Out[∘]= $\{W_1[1,1]\,W_2[1,1]\,W_3[1,1],\ W_1[1,1]\,W_2[1,1]\,W_3[1,2],$ … 46653 … $, W_1[6,6]\,W_2[6,6]\,W_3[6,6]\}$

large output | show less | show more | show all | set size limit...

In[∘]:= `(b // m_{1,2→1} // m_{1,3→1})`

Out[∘]= $\{W_1[1,1], W_1[1,2], W_1[1,3], W_1[1,4], W_1[1,5],$ … 46646 … $, W_1[6,2], W_1[6,3], W_1[6,4], W_1[6,5], W_1[6,6]\}$

large output | show less | show more | show all | set size limit...

η is a unit:

In[∘]:= `b = Basis[1]; (b // η_2 // m_{1,2→1}) == b == (b // η_2 // m_{2,1→1})`

Out[∘]= True

Δ is co-associative:

In[∘]:= `b = Basis[1]; (b // Δ_{1→1,2} // Δ_{2→2,3}) == (b // Δ_{1→1,3} // Δ_{1→1,2})`

Out[∘]= True

ε is a co-unit:

```
In[∘]:= b = Basis[1]; (b // Δ1→1,2 // ε2) == b == (b // Δ1→2,1 // ε2)
Out[∘]= True
```

ε is an algebra morphism:

```
In[∘]:= b = Basis[1, 2]; (b // ε1 // ε2) == (b // m1,2→1 // ε1)
Out[∘]= True
```

m is an algebra morphism:

```
In[∘]:= b = Basis[1, 3]; (b // Δ1→1,2 // Δ3→3,4 // m1,3→1 // m2,4→2) == (b // m1,3→1 // Δ1→1,2)
Out[∘]= True
```

S is an algebra anti-morphism:

```
In[∘]:= b = Basis[1, 2]; (b // m1,2→1 // S1) == (b // S1 // S2 // m2,1→1)
Out[∘]= True
```

S is a co-algebra anti-morphism:

```
In[∘]:= b = Basis[1]; (b // S1 // Δ1→1,2) == (b // Δ1→2,1 // S1 // S2)
Out[∘]= True
```

S is a convolution inverse of the identity:

```
In[∘]:= b = Basis[1]; (b // Δ1→1,2 // S2 // m1,2→1) == (b // ε1 // η1) == (b // Δ1→1,2 // S1 // m1,2→1)
Out[∘]= True
```

S is involutive:

```
In[∘]:= b = Basis[1]; (b // S1 // S1) == b
Out[∘]= True
```

Reidemeister 2:

```
In[∘]:= (R1,2 R̄3,4 // m1,3→1 // m2,4→2) == (1 // η1 // η2) == (R1,2 R̄3,4 // m1,3→1 // m4,2→2)
Out[∘]= True
```

Reidemeister 3:

```
In[∘]:= (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) == (R2,3 R1,4 R5,6 // m1,5→1 // m2,6→2 // m3,4→3)
Out[∘]= True
```

Compatibility of *R* and (Δ, *m*):

```
In[∘]:= {(R1,3 // Δ1→1,2 ) == ( R2,3 R1,4 // m3,4→3), (R1,2 // Δ2→2,3 ) == ( R0,2 R1,3 // m0,1→1)}
Out[∘]= {True, True}
```

Compatibility of *R* and ε:

In[∘]:= $\{(\mathbf{R}_{1,2}\ //\ \varepsilon_1) == (\mathbf{1}\ //\ \eta_2),\ (\mathbf{R}_{1,2}\ //\ \varepsilon_2) == (\mathbf{1}\ //\ \eta_1)\}$

Out[∘]= {True, True}

Compatibility of *R* and *S*:

In[∘]:= $(\mathbf{R}_{1,2}\ //\ \mathbf{S}_1) == \overline{\mathbf{R}}_{1,2} == (\mathbf{R}_{1,2}\ //\ \mathbf{S}_2)$

Out[∘]= True

Does R1 hold?

In[∘]:= $\{\mathbf{R}_{1,2}\ //\ \mathbf{m}_{1,2\to 1},\ \mathbf{1}\ //\ \eta_1\}$

Out[∘]= $\{W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + W_1[6, 6],$
$W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1]\}$

The forgotten relation, R1':

In[∘]:= $\left(\mathbf{Expand}\left[\mathbf{R}_{1,2}\,\overline{\mathbf{R}}_{4,3}\right]\ //\ \mathbf{m}_{1,2\to 1}\ //\ \mathbf{m}_{1,3\to 1}\ //\ \mathbf{m}_{1,4\to 1}\right) == (\mathbf{1}\ //\ \eta_1)$

Out[∘]= True

Computing the Knot Invariant for S_3

In[∘]:=
```
?? PositiveQ
```

Symbol

PositiveQ[xing] returns True if xing is a positive (right handed) crossing and False if it is negative (left handed).

Definitions

Out[∘]=
```
PositiveQ[X[KnotTheory`Private`i_,
    KnotTheory`Private`j_, KnotTheory`Private`k_, KnotTheory`Private`l_]] /;
  KnotTheory`Private`i == KnotTheory`Private`j || KnotTheory`Private`k ==
    KnotTheory`Private`l || KnotTheory`Private`j - KnotTheory`Private`l == 1 ||
   KnotTheory`Private`l - KnotTheory`Private`j > 1 = True

PositiveQ[X[KnotTheory`Private`i_,
    KnotTheory`Private`j_, KnotTheory`Private`k_, KnotTheory`Private`l_]] /;
  KnotTheory`Private`i == KnotTheory`Private`l || KnotTheory`Private`j ==
    KnotTheory`Private`k || KnotTheory`Private`l - KnotTheory`Private`j == 1 ||
   KnotTheory`Private`j - KnotTheory`Private`l > 1 = False

PositiveQ[_Xp] = True

PositiveQ[_Xm] = False
```

Full Name KnotTheory`PositiveQ

In[∘]:=
```
Z1[K_] := Z1[PD[K]];
Z1[pd_PD] := Module[{z},
  z = Expand[Times @@ pd /. x : X[i_, j_, k_, l_] :> If[PositiveQ@x, R_{l,i}, R̄_{j,i}]];
  Do[z = z // m_{1,k→1}, {k, 2, 2 Length@pd}];
  z]
```

In[∘]:=
```
tab1 = Table[K → Echo[Timing[Z1[K]]], {K, AllKnots[{3, 5}]}]
```

» {0.40625, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]}

» {14.8125, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}

Out[◦]= $Aborted

Edge-vertex convention: an oriented edge carries the same label as the vertex ending it.

```
In[◦]:= Z2[K_] := Z2[PD@K];
Z2[pd_PD] := Module[{z, done, st, c, mn, k},
  z = 1; done = {}; st = Range[2 Length@pd];
  Do[
   z *= c /. X[i_, j_, _, l_] :→ If[PositiveQ@c, mn = {i, l}; R_{l,i}, mn = {i, j};
       R̄_{j,i}];
   Do[
    If[MemberQ[done, k + 1], z = z // m_{k,k+1→k}; st = st /. k + 1 → k];
    If[MemberQ[done, k - 1], z = z // m_{st⟦k-1⟧,k→st⟦k-1⟧}; st = st /. k → st⟦k - 1⟧],
    {k, mn}];
   done = done ⋃ mn,
   {c, List @@ pd}];
  z]
```

```
In[◦]:= tab2 = Table[K → Echo[Timing[Z2[K]]], {K, AllKnots[{3, 6}]}]
```

» {0.109375, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]}

» {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}

» {5.89063, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}

» {0.9375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}

» {1.32813, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]}

» {0.984375, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]}

» {1.60938, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}

```
Out[◦]= {Knot[3, 1] → {0.109375, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]},
  Knot[4, 1] → {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
  Knot[5, 1] → {5.89063, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[5, 2] → {0.9375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[6, 1] → {1.32813, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]},
  Knot[6, 2] → {0.984375, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]},
  Knot[6, 3] → {1.60938, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}}
```

In[∘]:=
```
ThinPosition[K_] := Module[{todo, done, pd, c},
  todo = List @@ PD@K; done = {}; pd = PD[];
  While[todo =!= {},
   AppendTo[pd, c = RandomChoice@MaximalBy[todo, Length[done ⋂ List @@ #] &]];
   todo = DeleteCases[todo, c];
   done = done ⋃ List @@ c];
  pd
 ]
```

In[∘]:=
```
Z3[K_] := Z2@ThinPosition@K;
```

In[∘]:=
```
tab3 = Table[K → Echo[Timing[Z3[K]]], {K, AllKnots[{3, 7}]}]
```

```
» {0.109375, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]}
» {0.046875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
» {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.046875, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.0625, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]}
» {0.046875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]}
» {0.1875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
» {0.4375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.265625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.3125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.078125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}
» {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 1] + W1[5, 1] + W1[6, 6]}
» {2.21875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}
```

Out[∘]=
```
{Knot[3, 1] → {0.109375, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]},
 Knot[4, 1] → {0.046875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
 Knot[5, 1] → {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
 Knot[5, 2] → {0.046875, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
 Knot[6, 1] → {0.0625, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]},
 Knot[6, 2] → {0.046875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]},
 Knot[6, 3] → {0.1875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
 Knot[7, 1] → {0.4375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
 Knot[7, 2] → {0.265625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
 Knot[7, 3] → {0.3125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
 Knot[7, 4] → {0.078125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]},
 Knot[7, 5] → {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
 Knot[7, 6] → {0.375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 1] + W1[5, 1] + W1[6, 6]},
 Knot[7, 7] → {2.21875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}}
```

Computing the Knot Invariant for S_4

In[∘]:=
```
DeclareGroup[S_4];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[∘]//MatrixForm=

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
8	7	14	13	20	19	2	1	16	15	22	21	4	3	10	9	24	23	6	5	12	11	18	17
9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
13	19	7	20	8	14	15	21	1	22	2	16	9	23	3	24	4	10	11	17	5	18	6	12
14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
20	14	19	8	13	7	22	16	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	7	15	1	9	3
23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

In[∘]:=
```
Table[K → Echo[Timing[Z3[K]]], {K, AllKnots[{3, 7}]}]
```

»
```
{4.98438, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + 4 W_1[4, 1] + 4 W_1[5, 1] +
  5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] +
  4 W_1[11, 14] + 4 W_1[12, 1] + 4 W_1[13, 1] + 4 W_1[14, 11] + W_1[14, 14] + 5 W_1[15, 15] +
  4 W_1[16, 1] + W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] +
  4 W_1[20, 1] + 4 W_1[21, 1] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]}
```

Out[∘]= $Aborted

Computing the knot invariant for S_4 by summing over conjugacy classes

In[∘]:=
```
DeclareGroup[S_4];
```

pdf

In[∘]:=
```
ConjugacyClasses := Module[{sea = Range@n, ccs = {}, cc},
  While[Length@sea > 0,
   cc = Union[Table[m[inv[α], First@sea, α], {α, n}]];
   AppendTo[ccs, cc];
   sea = Complement[sea, cc];
  ]; ccs]
```

In[∘]:=
```
CC = ConjugacyClasses
```

Out[∘]=
```
{{1}, {2, 3, 6, 7, 15, 22}, {4, 5, 9, 12, 13, 16, 20, 21}, {8, 17, 24}, {10, 11, 14, 18, 19, 23}}
```

pdf

In[∘]:=
```
(* $CIS is a Conjugation Invariant Set *)
R_{i_,j_} := Sum[W_i[α, 1] W_j[β, α], {α, $CIS}, {β, $CIS}];
R̄_{i_,j_} := Sum[W_i[α, 1] W_j[β, inv@α], {α, $CIS}, {β, $CIS}];
η_{i_}[ℰ_] := Expand@Sum[ℰ W_i[α, 1], {α, $CIS}];
```

Reidemeister 2:

In[∘]:=
```
Table[
 (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) == (1 // η_1 // η_2) == (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{4,2→2}),
 {$CIS, CC} ]
```

Out[∘]=
```
{True, True, True, True, True}
```

Reidemeister 3:

In[∘]:=
```
Table[
 (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) == (R_{2,3} R_{1,4} R_{5,6} // m_{1,5→1} // m_{2,6→2} // m_{3,4→3}),
 {$CIS, CC} ]
```

Out[∘]=
```
{True, True, True, True, True}
```

In[∘]:=
```
Z4[K_] := Z4[PD@K];
Z4[pd_PD] := Sum[
   Module[{z, done, st, c, mn, k},
    z = 1; done = {}; st = Range[2 Length@pd];
    Do[
     z *= c /. X[i_, j_, _, l_] :> If[PositiveQ@c, mn = {i, l}; R_{l,i}, mn = {i, j};
        R̄_{j,i}];
     Do[
      If[MemberQ[done, k + 1], z = z // m_{k,k+1→k}; st = st /. k + 1 → k];
      If[MemberQ[done, k - 1], z = z // m_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]]],
      {k, mn}];
     done = done ⋃ mn,
     {c, List @@ pd}];
    z ],
   {$CIS, CC} ];
Z5[K_] := Z4@ThinPosition@K;
```

In[∘]:=
```
Table[K → Echo[Timing[Z5[K]]], {K, AllKnots[{3, 7}]}]
```

» {0.125, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + 4 W_1[4, 1] + 4 W_1[5, 1] + 5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + 4 W_1[12, 1] + 4 W_1[13, 1] + 4 W_1[14, 11] + W_1[14, 14] + 5 W_1[15, 15] + 4 W_1[16, 1] + W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] + 4 W_1[20, 1] + 4 W_1[21, 1] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]}

» {1.04688, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + 4 W_1[4, 1] + 4 W_1[5, 1] + W_1[6, 1] + W_1[7, 1] + W_1[8, 1] + 4 W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + 4 W_1[12, 1] + 4 W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + 4 W_1[16, 1] + W_1[17, 1] + W_1[18, 1] + W_1[19, 1] + 4 W_1[20, 1] + 4 W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + W_1[24, 1]}

» {0.203125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] + W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]}

» {1.03125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] + W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]}

» {1.1875, W_1[1, 1] + 5 W_1[2, 1] + 5 W_1[3, 1] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 1] + 5 W_1[7, 1] + W_1[8, 1] + W_1[9, 9] + 4 W_1[10, 1] + W_1[10, 17] + 4 W_1[11, 1] + W_1[11, 24] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 1] + W_1[14, 24] + 5 W_1[15, 1] + W_1[16, 16] + W_1[17, 1] + 4 W_1[18, 1] + W_1[18, 8] + 4 W_1[19, 1] + W_1[19, 17] + W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 1] + 4 W_1[23, 1] + W_1[23, 8] + W_1[24, 1]}

» {0.75, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 4] + W_1[5, 5] + W_1[6, 1] + W_1[7, 1] + W_1[8, 1] + W_1[9, 9] + W_1[10, 17] + W_1[11, 24] + W_1[12, 12] + W_1[13, 13] + W_1[14, 24] + W_1[15, 1] + W_1[16, 16] + W_1[17, 1] + W_1[18, 8] + W_1[19, 17] + W_1[20, 20] + W_1[21, 21] + W_1[22, 1] + W_1[23, 8] + W_1[24, 1]}

» {1.17188, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1] + W_1[7, 1] + W_1[8, 1] + W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + W_1[12, 1] + W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + W_1[16, 1] + W_1[17, 1] + W_1[18, 1] + W_1[19, 1] + W_1[20, 1] + W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + W_1[24, 1]}

» $\{0.84375, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] +$
$W_1[10, 10] + W_1[11, 11] + W_1[12, 20] + W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] +$
$W_1[18, 18] + W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{1.14063, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 5] + 4 W_1[5, 4] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 13] + W_1[10, 10] + W_1[11, 11] + 4 W_1[12, 20] +$
$4 W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + 4 W_1[16, 21] + W_1[17, 17] + W_1[18, 18] +$
$W_1[19, 19] + 4 W_1[20, 12] + 4 W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{0.34375, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 4] + 4 W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + 4 W_1[12, 12] +$
$4 W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + 4 W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + 4 W_1[20, 20] + 4 W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\}$

» $\{0.71875, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] +$
$W_1[9, 9] + 4 W_1[10, 10] + W_1[10, 19] + 4 W_1[11, 11] + W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + W_1[14, 11] +$
$4 W_1[14, 14] + 5 W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + 4 W_1[18, 18] + W_1[18, 23] + W_1[19, 10] +$
$4 W_1[19, 19] + W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + W_1[23, 18] + 4 W_1[23, 23] + W_1[24, 24]\}$

» $\{1.15625, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] +$
$W_1[10, 10] + W_1[11, 11] + W_1[12, 20] + W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] +$
$W_1[18, 18] + W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{2.23438, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 1] + W_1[5, 1] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] +$
$W_1[9, 1] + W_1[10, 10] + W_1[11, 11] + W_1[12, 1] + W_1[13, 1] + W_1[14, 14] + W_1[15, 15] + W_1[16, 1] +$
$W_1[17, 17] + W_1[18, 18] + W_1[19, 19] + W_1[20, 1] + W_1[21, 1] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{2., W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] + W_1[9, 9] +$
$W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 11] +$
$W_1[14, 14] + 5 W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] +$
$W_1[19, 19] + W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]\}$

Out[∘]= $\{\text{Knot}[3, 1] \to$
$\{0.125, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + 4 W_1[4, 1] + 4 W_1[5, 1] + 5 W_1[6, 6] + 5 W_1[7, 7] +$
$W_1[8, 8] + 4 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + 4 W_1[12, 1] +$
$4 W_1[13, 1] + 4 W_1[14, 11] + W_1[14, 14] + 5 W_1[15, 15] + 4 W_1[16, 1] +$
$W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] + 4 W_1[20, 1] +$
$4 W_1[21, 1] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]\},$
$\text{Knot}[4, 1] \to \{1.04688, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + 4 W_1[4, 1] + 4 W_1[5, 1] +$
$W_1[6, 1] + W_1[7, 1] + W_1[8, 1] + 4 W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + 4 W_1[12, 1] +$
$4 W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + 4 W_1[16, 1] + W_1[17, 1] + W_1[18, 1] +$
$W_1[19, 1] + 4 W_1[20, 1] + 4 W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + W_1[24, 1]\},$
$\text{Knot}[5, 1] \to \{0.203125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] +$
$W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\},$
$\text{Knot}[5, 2] \to \{1.03125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] +$
$W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\},$
$\text{Knot}[6, 1] \to \{1.1875, W_1[1, 1] + 5 W_1[2, 1] + 5 W_1[3, 1] + W_1[4, 4] + W_1[5, 5] +$
$5 W_1[6, 1] + 5 W_1[7, 1] + W_1[8, 1] + W_1[9, 9] + 4 W_1[10, 1] + W_1[10, 17] + 4 W_1[11, 1] +$
$W_1[11, 24] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 1] + W_1[14, 24] + 5 W_1[15, 1] +$
$W_1[16, 16] + W_1[17, 1] + 4 W_1[18, 1] + W_1[18, 8] + 4 W_1[19, 1] + W_1[19, 17] +$

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    W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 1] + 4 W_1[23, 1] + W_1[23, 8] + W_1[24, 1]},
  Knot[6, 2] → {0.75, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 4] + W_1[5, 5] + W_1[6, 1] +
    W_1[7, 1] + W_1[8, 1] + W_1[9, 9] + W_1[10, 17] + W_1[11, 24] + W_1[12, 12] + W_1[13, 13] +
    W_1[14, 24] + W_1[15, 1] + W_1[16, 16] + W_1[17, 1] + W_1[18, 8] + W_1[19, 17] +
    W_1[20, 20] + W_1[21, 21] + W_1[22, 1] + W_1[23, 8] + W_1[24, 1]}, Knot[6, 3] →
   {1.17188, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1] + W_1[7, 1] + W_1[8, 1] +
    W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + W_1[12, 1] + W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + W_1[16, 1] +
    W_1[17, 1] + W_1[18, 1] + W_1[19, 1] + W_1[20, 1] + W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + W_1[24, 1]},
  Knot[7, 1] → {0.84375, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] +
    W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] + W_1[10, 10] + W_1[11, 11] + W_1[12, 20] +
    W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] + W_1[18, 18] +
    W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]},
  Knot[7, 2] → {1.14063, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 5] + 4 W_1[5, 4] +
    W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 13] + W_1[10, 10] + W_1[11, 11] + 4 W_1[12, 20] +
    4 W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + 4 W_1[16, 21] + W_1[17, 17] + W_1[18, 18] +
    W_1[19, 19] + 4 W_1[20, 12] + 4 W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]},
  Knot[7, 3] → {0.34375, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 4] + 4 W_1[5, 5] +
    W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + 4 W_1[12, 12] +
    4 W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + 4 W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +
    W_1[19, 10] + 4 W_1[20, 20] + 4 W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]},
  Knot[7, 4] → {0.71875, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] +
    5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + 4 W_1[10, 10] + W_1[10, 19] + 4 W_1[11, 11] +
    W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + W_1[14, 11] + 4 W_1[14, 14] + 5 W_1[15, 15] +
    W_1[16, 16] + W_1[17, 17] + 4 W_1[18, 18] + W_1[18, 23] + W_1[19, 10] + 4 W_1[19, 19] +
    W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + W_1[23, 18] + 4 W_1[23, 23] + W_1[24, 24]},
  Knot[7, 5] → {1.15625, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] +
    W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] + W_1[10, 10] + W_1[11, 11] + W_1[12, 20] +
    W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] + W_1[18, 18] +
    W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]},
  Knot[7, 6] → {2.23438, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 1] + W_1[5, 1] +
    W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 1] + W_1[10, 10] + W_1[11, 11] + W_1[12, 1] +
    W_1[13, 1] + W_1[14, 14] + W_1[15, 15] + W_1[16, 1] + W_1[17, 17] + W_1[18, 18] +
    W_1[19, 19] + W_1[20, 1] + W_1[21, 1] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]},
  Knot[7, 7] → {2., W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 6] +
    5 W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] +
    4 W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 11] + W_1[14, 14] + 5 W_1[15, 15] +
    W_1[16, 16] + W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] +
    W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]}}
```

Computing the knot invariant for S_5 by summing over conjugacy classes

In[∘]:=
```
DeclareGroup[S_5];
CC = ConjugacyClasses
```

```
Out[ ]= {{1}, {2, 3, 6, 7, 15, 22, 25, 55, 81, 106},
  {4, 5, 9, 12, 13, 16, 20, 21, 31, 39, 46, 49, 57, 60, 75, 79, 82, 100, 104, 105},
  {8, 17, 24, 26, 27, 30, 56, 61, 68, 83, 87, 95, 108, 112, 120}, {10, 11, 14, 18, 19, 23, 33, 36,
   37, 40, 44, 45, 51, 54, 58, 59, 63, 70, 73, 76, 80, 84, 85, 96, 98, 99, 103, 107, 110, 119},
  {28, 29, 32, 41, 48, 50, 62, 66, 67, 71, 77, 88, 89, 92, 93, 102, 111, 114, 115, 118}, {34, 35,
   38, 42, 43, 47, 52, 53, 64, 65, 69, 72, 74, 78, 86, 90, 91, 94, 97, 101, 109, 113, 116, 117}}

In[ ]:= Table[K → Echo[Timing[Z5[K]]], {K, AllKnots[{3, 7}]}]
```

» $\{20.5, W_1[1, 1] + 7 W_1[2, 2] + 7 W_1[3, 3] + 7 W_1[4, 1] + 7 W_1[5, 1] + 7 W_1[6, 6] + 7 W_1[7, 7] + 5 W_1[8, 8] +$
$7 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + 7 W_1[12, 1] + 7 W_1[13, 1] +$
$4 W_1[14, 11] + W_1[14, 14] + 7 W_1[15, 15] + 7 W_1[16, 1] + 5 W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] +$
$4 W_1[19, 10] + W_1[19, 19] + 7 W_1[20, 1] + 7 W_1[21, 1] + 7 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] +$
$5 W_1[24, 24] + 7 W_1[25, 25] + 5 W_1[26, 26] + 5 W_1[27, 27] + W_1[28, 25] + W_1[29, 25] + 5 W_1[30, 30] +$
$7 W_1[31, 1] + W_1[32, 2] + W_1[33, 33] + 4 W_1[33, 73] + W_1[34, 65] + 5 W_1[34, 91] + W_1[35, 72] +$
$5 W_1[35, 116] + W_1[36, 36] + 4 W_1[36, 98] + W_1[37, 37] + 4 W_1[37, 51] + W_1[38, 94] + 5 W_1[38, 113] +$
$7 W_1[39, 1] + W_1[40, 40] + 4 W_1[40, 99] + W_1[41, 6] + 5 W_1[42, 69] + W_1[42, 86] + 5 W_1[43, 90] +$
$W_1[43, 117] + W_1[44, 44] + 4 W_1[44, 54] + W_1[45, 45] + 4 W_1[45, 76] + 7 W_1[46, 1] + 5 W_1[47, 64] +$
$W_1[47, 109] + W_1[48, 3] + 7 W_1[49, 1] + W_1[50, 2] + 4 W_1[51, 37] + W_1[51, 51] + W_1[52, 90] +$
$5 W_1[52, 117] + 5 W_1[53, 94] + W_1[53, 113] + 4 W_1[54, 44] + W_1[54, 54] + 7 W_1[55, 55] + 5 W_1[56, 56] +$
$7 W_1[57, 1] + W_1[58, 58] + 4 W_1[58, 103] + W_1[59, 59] + 4 W_1[59, 80] + 7 W_1[60, 1] + 5 W_1[61, 61] +$
$W_1[62, 55] + W_1[63, 63] + 4 W_1[63, 85] + W_1[64, 47] + 5 W_1[64, 78] + 5 W_1[65, 34] + W_1[65, 97] +$
$W_1[66, 15] + W_1[67, 55] + 5 W_1[68, 68] + W_1[69, 42] + 5 W_1[69, 101] + W_1[70, 70] + 4 W_1[70, 110] +$
$W_1[71, 22] + 5 W_1[72, 35] + W_1[72, 74] + 4 W_1[73, 33] + W_1[73, 73] + 5 W_1[74, 72] + W_1[74, 116] +$
$7 W_1[75, 1] + 4 W_1[76, 45] + W_1[76, 76] + W_1[77, 6] + W_1[78, 64] + 5 W_1[78, 109] + 7 W_1[79, 1] +$
$4 W_1[80, 59] + W_1[80, 80] + 7 W_1[81, 81] + 7 W_1[82, 1] + 5 W_1[83, 83] + W_1[84, 84] + 4 W_1[84, 107] +$
$4 W_1[85, 63] + W_1[85, 85] + 5 W_1[86, 42] + W_1[86, 101] + 5 W_1[87, 87] + W_1[88, 7] + W_1[89, 81] +$
$W_1[90, 43] + 5 W_1[90, 52] + W_1[91, 34] + 5 W_1[91, 97] + W_1[92, 22] + W_1[93, 81] + 5 W_1[94, 38] +$
$W_1[94, 53] + 5 W_1[95, 95] + W_1[96, 96] + 4 W_1[96, 119] + 5 W_1[97, 65] + W_1[97, 91] + 4 W_1[98, 36] +$
$W_1[98, 98] + 4 W_1[99, 40] + W_1[99, 99] + 7 W_1[100, 1] + W_1[101, 69] + 5 W_1[101, 86] + W_1[102, 3] +$
$4 W_1[103, 58] + W_1[103, 103] + 7 W_1[104, 1] + 7 W_1[105, 1] + 7 W_1[106, 106] + 4 W_1[107, 84] +$
$W_1[107, 107] + 5 W_1[108, 108] + 5 W_1[109, 47] + W_1[109, 78] + 4 W_1[110, 70] + W_1[110, 110] + W_1[111, 7] +$
$5 W_1[112, 112] + W_1[113, 38] + 5 W_1[113, 53] + W_1[114, 106] + W_1[115, 15] + W_1[116, 35] + 5 W_1[116, 74] +$
$5 W_1[117, 43] + W_1[117, 52] + W_1[118, 106] + 4 W_1[119, 96] + W_1[119, 119] + 5 W_1[120, 120]\}$

» $\{38.9531, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + 7 W_1[4, 1] + 7 W_1[5, 1] + W_1[6, 1] + W_1[7, 1] + 9 W_1[8, 1] +$
$7 W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + 7 W_1[12, 1] + 7 W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + 7 W_1[16, 1] +$
$9 W_1[17, 1] + W_1[18, 1] + W_1[19, 1] + 7 W_1[20, 1] + 7 W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + 9 W_1[24, 1] +$
$W_1[25, 1] + 9 W_1[26, 1] + 9 W_1[27, 1] + W_1[28, 1] + 6 W_1[28, 4] + 6 W_1[28, 5] + W_1[29, 1] + 6 W_1[29, 4] +$
$6 W_1[29, 5] + 9 W_1[30, 1] + 7 W_1[31, 1] + W_1[32, 1] + 6 W_1[32, 31] + 6 W_1[32, 49] + W_1[33, 1] +$
$W_1[34, 1] + W_1[35, 1] + W_1[36, 1] + W_1[37, 1] + W_1[38, 1] + 7 W_1[39, 1] + W_1[40, 1] + W_1[41, 1] +$
$6 W_1[41, 39] + 6 W_1[41, 75] + W_1[42, 1] + W_1[43, 1] + W_1[44, 1] + W_1[45, 1] + 7 W_1[46, 1] + W_1[47, 1] +$
$W_1[48, 1] + 6 W_1[48, 46] + 6 W_1[48, 100] + 7 W_1[49, 1] + W_1[50, 1] + 6 W_1[50, 31] + 6 W_1[50, 49] +$
$W_1[51, 1] + W_1[52, 1] + W_1[53, 1] + W_1[54, 1] + W_1[55, 1] + 9 W_1[56, 1] + 7 W_1[57, 1] + W_1[58, 1] +$
$W_1[59, 1] + 7 W_1[60, 1] + 9 W_1[61, 1] + W_1[62, 1] + 6 W_1[62, 16] + 6 W_1[62, 21] + W_1[63, 1] +$
$W_1[64, 1] + W_1[65, 1] + W_1[66, 1] + 6 W_1[66, 60] + 6 W_1[66, 104] + W_1[67, 1] + 6 W_1[67, 16] +$
$6 W_1[67, 21] + 9 W_1[68, 1] + W_1[69, 1] + W_1[70, 1] + W_1[71, 1] + 6 W_1[71, 57] + 6 W_1[71, 79] +$
$W_1[72, 1] + W_1[73, 1] + W_1[74, 1] + 7 W_1[75, 1] + W_1[76, 1] + W_1[77, 1] + 6 W_1[77, 39] +$
$6 W_1[77, 75] + W_1[78, 1] + 7 W_1[79, 1] + W_1[80, 1] + W_1[81, 1] + 7 W_1[82, 1] + 9 W_1[83, 1] +$
$W_1[84, 1] + W_1[85, 1] + W_1[86, 1] + 9 W_1[87, 1] + W_1[88, 1] + 6 W_1[88, 82] + 6 W_1[88, 105] +$
$W_1[89, 1] + 6 W_1[89, 12] + 6 W_1[89, 20] + W_1[90, 1] + W_1[91, 1] + W_1[92, 1] + 6 W_1[92, 57] +$
$6 W_1[92, 79] + W_1[93, 1] + 6 W_1[93, 12] + 6 W_1[93, 20] + W_1[94, 1] + 9 W_1[95, 1] + W_1[96, 1] +$
$W_1[97, 1] + W_1[98, 1] + W_1[99, 1] + 7 W_1[100, 1] + W_1[101, 1] + W_1[102, 1] + 6 W_1[102, 46] +$
$6 W_1[102, 100] + W_1[103, 1] + 7 W_1[104, 1] + 7 W_1[105, 1] + W_1[106, 1] + W_1[107, 1] + 9 W_1[108, 1] +$
$W_1[109, 1] + W_1[110, 1] + W_1[111, 1] + 6 W_1[111, 82] + 6 W_1[111, 105] + 9 W_1[112, 1] + W_1[113, 1] +$
$W_1[114, 1] + 6 W_1[114, 9] + 6 W_1[114, 13] + W_1[115, 1] + 6 W_1[115, 60] + 6 W_1[115, 104] +$
$W_1[116, 1] + W_1[117, 1] + W_1[118, 1] + 6 W_1[118, 9] + 6 W_1[118, 13] + W_1[119, 1] + 9 W_1[120, 1]\}$

Out[ ]= $Aborted

Conclusion

A very elegant theory which we implemented cleanly and were able to improve dramatically.

Overall, a great failure.