Pensieve header: The Kerler Algebra and the Alexander polynomial. Closely related to arXiv://math-/0008204.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory"];
<< Common.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=

MT =

	a	b	c	d	ka	kb	kc	kd
a	a	b	0	0	ka	kb	0	0
b	0	0	a	b	0	0	-ka	-kb
c	c	d	0	0	-kc	-kd	0	0
d	0	0	c	d	0	0	kc	kd
ka	ka	kb	0	0	a	b	0	0
kb	0	0	ka	kb	0	0	-a	-b
kc	kc	kd	0	0	-c	-d	0	0
kd	0	0	kc	kd	0	0	c	d

;

$\mathcal{E}$_ // $m_{i\_,j\_\to k\_}$:= Expand[$\mathcal{E}$] /. Flatten@Table[$MT[\![\alpha, 1]\!]_i\, MT[\![1, \beta]\!]_j \to$ $(MT[\![\alpha, \beta]\!]$ /. v : (a | b | c | d | ka | kb | kc | kd) :→ v_k), {α, 2, 9}, {β, 2, 9}];

$\eta_{i\_}$:=
 a_i +
 d_i;

In[∘]:=
```
KBasis[i_] := {a_i, b_i, c_i, d_i, ka_i, kb_i, kc_i, kd_i};
KBasis[i_, is__] := Flatten@Outer[Times, KBasis[i], KBasis[is]]
```

Associativity of *m*:

In[∘]:=
Short[lhs = KBasis[1, 2, 3] // $m_{1,2\to1}$ // $m_{1,3\to1}$]
rhs = KBasis[1, 2, 3] // $m_{2,3\to2}$ // $m_{1,2\to1}$!;
lhs == rhs

Out[∘]//Short= {a_1, b_1, 0, 0, ka_1, kb_1, 0, 0, 0, 0, a_1, b_1, 0, 0, <<484>>, 0, 0, $-kc_1$, $-kd_1$, 0, 0, 0, 0, c_1, d_1, 0, 0, kc_1, kd_1}

Out[∘]= True

Units:

In[∘]:= {(η_1 KBasis[2] // $m_{1,2\to1}$) == KBasis[1], (η_1 KBasis[2] // $m_{2,1\to1}$) == KBasis[1]}

Out[∘]= {True, True}

In[∘]:=

$R_{i\_,j\_}$:= $T^{-1/2}\left(a_i\, ka_j - T\, a_i\, kd_j + (T-1)\, b_j\, kc_i + d_i\, ka_j + T\, d_i\, kd_j\right)$;

$\overline{R}_{i\_,j\_}$:= $T^{1/2}\left(a_i\, ka_j - T^{-1}\, a_i\, kd_j + \left(T^{-1}-1\right) b_j\, kc_i + d_i\, ka_j + T^{-1}\, d_i\, kd_j\right)$;

$C_{i\_}$:= $T^{-1/2}\,(ka_i + kd_i)$; $\overline{C}_{i\_}$:= $T^{1/2}\,(ka_i + kd_i)$;

Reidemeister 3:

In[∘]:=
```
Short[lhs = R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3]
rhs = R2,3 R1,4 R5,6 // m1,5→1 // m2,6→2 // m3,4→3;
lhs == rhs
```

Out[∘]//Short= $\frac{a_1\,a_3\,ka_2}{T^{3/2}}+\frac{a_3\,d_1\,ka_2}{T^{3/2}}+\ll 23\gg+\sqrt{T}\,kb_3\,kc_1\,kd_2-T^{3/2}\,kb_3\,kc_1\,kd_2$

Out[∘]= True

Reidemeister 2b:

In[∘]:=
```
Short[lhs = R1,2 R̄3,4 // m1,3→1 // m2,4→2]
rhs = η1 η2 // Expand;
lhs == rhs
```

Out[∘]//Short= $a_1\,a_2+a_2\,d_1+a_1\,d_2+d_1\,d_2$

Out[∘]= True

Naive Reidemeister 2c:

In[∘]:=
```
Short[lhs = R1,2 R̄3,4 // m1,3→1 // m4,2→2]
rhs = η1 η2 // Expand;
Simplify[lhs - rhs]
```

Out[∘]//Short= $a_1\,a_2+a_2\,d_1+a_1\,d_2+d_1\,d_2-2\,kb_2\,kc_1+2\,T\,kb_2\,kc_1$

Out[∘]= $2\times(-1+T)\,kb_2\,kc_1$

Corrected Reidemeister 2c:

In[∘]:=
```
lhs = R1,4 R̄5,2 C3 // m2,4→2 // m1,3→1 // m1,5→1
rhs = C1 η2 // Expand;
lhs == rhs
```

Out[∘]= $\frac{a_2\,ka_1}{\sqrt{T}}+\frac{d_2\,ka_1}{\sqrt{T}}+\frac{a_2\,kd_1}{\sqrt{T}}+\frac{d_2\,kd_1}{\sqrt{T}}$

Out[∘]= True

$C\,\overline{C}$:

In[∘]:=
```
C1 C̄2 // m1,2→1
```

Out[∘]= a_1+d_1

Reidemeister 1s:

In[∘]:=
```
{(C̄2 R1,3 // m1,2→1 // m1,3→1) == η1, (C̄2 R̄3,1 // m1,2→1 // m1,3→1) == η1,
 (C2 R̄1,3 // m1,2→1 // m1,3→1) == η1, (C2 R3,1 // m1,2→1 // m1,3→1) == η1}
```

Out[∘]= {True, True, True, True}

The whirl:

In[∘]:=
```
Expand[(C̄₁ C̄₂ R₃,₄ C₅ C₆ // m₁,₃→₁ // m₁,₅→₁ // m₂,₄→₂ // m₂,₆→₂) == R₁,₂]
```

Out[∘]= True

RVK and Z

RVK, rot, Z modified from 2016-09/OneSmidgen.nb. See also in AP/Projects/SL2Invariant/.

Some details of the code below are at http://drorbn.net/bbs/show?shot=Dror-160920-151350.jpg.

In[∘]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
   xs and a length 2n list of rotation numbers rots. Crossing
   sites are indexed 1 through 2n, and rots⟦k⟧ is the rotation
   between site k-1 and site k. RVK is also a casting operator
   converting to the RVK presentation from other knot presentations.";
```

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :→ { Xp[x⟦4⟧, x⟦1⟧]  PositiveQ@x
                          Xm[x⟦2⟧, x⟦1⟧]  True       ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :→ {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :→ (++rots⟦l⟧; {1 - l, k + 1, l}),
            _Xp | _Xm :→ {}
           }), {1}],
     Cases[front, k | -k] /. {k, -k} :→ --rots⟦k + 1⟧;
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot_i_ [n_] := rot_i[n] = { η_i                           n == 0
                            C_$ rot_i[n - 1] // m_{i,$→i}  n > 0
                            C̄_$ rot_i[n + 1] // m_{i,$→i}  n < 0
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := Module[{ζ, done, st, c, χ, i, j, k},
  ζ = 1; done = {}; st = Range[2 Length[rvk[[1]]]];
  Do[
   {i, j} = List @@ c;
   χ = c /. {_Xp :→ R_{i,j}, _Xm :→ R̄_{i,j}};
   Do[χ = (rot_0[rvk[[2, k]]] χ) // m_{0,k→k}, {k, {i, j}}];
   ζ *= χ;
   Do[
    If[MemberQ[done, k + 1], ζ = ζ // m_{k,k+1→k}; st = st /. k + 1 → k];
    If[MemberQ[done, k - 1], ζ = ζ // m_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]]],
    {k, {i, j}}];
   done = done ⋃ {i, j},
   {c, rvk[[1]]}
  ];
  Factor@ζ
 ]
```

In[∘]:=
```
K = Knot[8, 17]; Factor@Alexander[K][T]
z = Z[K]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]= $-\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3}$

Out[∘]= $-\frac{\left(1-4T+8T^2-11T^3+8T^4-4T^5+T^6\right)(a_1+d_1)}{T^3}$

In[∘]:=
```
Timing@Union@Table[Simplify[Alexander[K][T] η_1 == Z[K]], {K, AllKnots[{3, 10}]}]
```

Out[∘]= {138.344, {True}}

In[∘]:=
```
ZF[K_] := Z@ThinPosition@K;
```

In[∘]:=
```
Timing@Union@Table[Simplify[Alexander[K][T] η_1 == ZF[K]], {K, AllKnots[{3, 10}]}]
```

Out[∘]= {20.7969, {True}}

In[∘]:=
```
Timing[ZF[GST48]]
```

Out[∘]= $\left\{63.75, -\frac{\left(-1+2T-T^2-T^3+2T^4-T^5+T^8\right)\times\left(-1+T^3-2T^4+T^5+T^6-2T^7+T^8\right)(a_1+d_1)}{T^8}\right\}$