

Perturbed Gaussian Integration and Feynman Diagrams

Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of \mathbb{R}^n , let $(t_i)_{i=1}^n$ be a set of “dual” variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

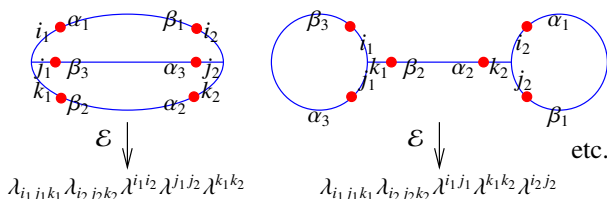
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k} = \sum_{m \geq 0} \frac{\epsilon^m}{6^m m!} \int_{\mathbb{R}^n} (\lambda_{ijk}x^i x^j x^k)^m e^{-\frac{1}{2}\lambda_{ij}x^i x^j}$$

Feynman 

$$= \sum_{m \geq 0} \frac{C \epsilon^m}{6^m m!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m e^{\frac{1}{2} \lambda^{\alpha\beta} t_\alpha t_\beta} \Big|_{t_\alpha=0} = \sum_{\substack{m, l \geq 0 \\ 3m = 2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m (\lambda^{\alpha\beta} t_\alpha t_\beta)^l$$

$$= \sum_{\substack{m, l \geq 0 \\ 3m = 2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \left[\begin{array}{c} \lambda^{\alpha_1 \beta_1} \quad \lambda^{\alpha_2 \beta_2} \quad \lambda^{\alpha_3 \beta_3} \quad \dots \quad \lambda^{\alpha_l \beta_l} \\ \text{---} t_{\alpha_1} \quad t_{\beta_1} \quad \text{---} t_{\alpha_2} \quad t_{\beta_2} \quad \text{---} t_{\alpha_3} \quad t_{\beta_3} \quad \dots \quad \text{---} t_{\alpha_l} \quad t_{\beta_l} \\ \dots \text{ sum over all pairings } \dots \\ \partial^{i_1} \quad \partial^{j_1} \quad \partial^{k_1} \quad \partial^{i_2} \quad \partial^{j_2} \quad \partial^{k_2} \quad \dots \quad \partial^{i_m} \quad \partial^{j_m} \quad \partial^{k_m} \\ \text{---} \lambda_{i_1 j_1 k_1} \quad \text{---} \lambda_{i_2 j_2 k_2} \quad \dots \quad \text{---} \lambda_{i_m j_m k_m} \end{array} \right]$$

$$= \sum_{\substack{m, l \geq 0 \\ 3m = 2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked} \\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$



$$= C \sum_{\text{unmarked Feynman diagrams } D} \frac{\epsilon^{m(D)} \mathcal{E}(D)}{|\text{Aut}(D)|}$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m! 2^l l!}{|\text{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \times S_m] \times [(S_2)^l \times S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D , and the stabilizer of any given P is $\text{Aut}(D)$. □

The Fourier Transform.

$(F: V \rightarrow \mathbb{C}) \Rightarrow (\tilde{f}: V^* \rightarrow \mathbb{C})$
via $\tilde{F}(\varphi) := \int_V f(v) e^{-i\langle \varphi, v \rangle} dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v) dv$.
- $\frac{\partial}{\partial \varphi_i} \tilde{f} \sim v^i \tilde{f}$.
- $(e^{Q/2}) \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v \rangle$ for $L: V \rightarrow V^*$, and $Q^{-1}(\varphi) := \langle \varphi, L^{-1}\varphi \rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.

