Tuesday Feb 24, hour 19: Quiz 6 and "Divide into Cases"

February-24-15 9:15 AM

Return Quiz 5?

Quiz 6.

APNS.

Discuss Quiz 6.

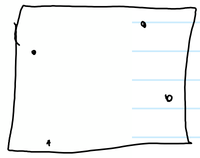
Anybody ready to claim the $50 bounty?

Any beautiful symmetries spotted?

2.5.11.

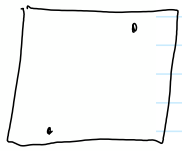
(a) Let R_n denote the number of ways of placing n nonattacking rooks on the n-by-n chessboard so that the arrangement is symmetric about a 90° clockwise rotation of the board about the center. Show that

$$R_{4n} = (4n-2)R_{4n-4},$$
$$R_{4n+1} = R_{4n},$$
$$R_{4n+2} = 0 = R_{4n+3}.$$



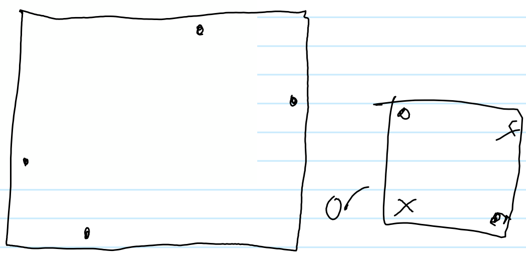
(b) Let S_n denote the number of ways of placing n nonattacking rooks on the n-by-n chessboard so that the arrangement is symmetric about the center of the board. Show that

$$S_{2n} = 2nS_{2n-2},$$
$$S_{2n+1} = S_{2n}.$$



(c) Let T_n denote the number of ways of placing n nonattacking rooks on the n-by-n chessboard so that the arrangement is symmetric about both diagonals. Show that

$$S_2 = 2,$$
$$S_{2n+1} = S_{2n},$$
$$S_{2n} = 2S_{2n-2} + (2n-2)S_{2n-4}.$$



Problem 2 (Larson's 2.5.13).

1. A *derangement* is a permutation $\sigma \in S_n$ such that for every i, $\sigma i \neq i$. Let g_n be the number of derangements in S_n. Show that

$$g_1 = 0, \quad g_2 = 1, \quad g_n = (n-1)(g_{n-1} + g_{n-2}).$$

Hint. A derangement interchanges 1 with some other element, or not.

2. Let f_n be the number of permutations in S_n that have exactly one fixed point (namely, exactly one i such that $\sigma i = i$). Show that $|f_n - g_n| = 1$.

$$f_n = n \cdot g_{n-1}$$

$$d_n = f_n - g_n = n g_{n-1} - (n-1) g_{n-1} - (n-1) g_{n-2}$$

$$= g_{n-1} - (n-1) g_{n-2} = -d_{n-1}$$

Problem 3 (Larson's 1.8.1, modified).

1. Let $0 < \alpha < \pi$. Show that $\frac{\sin\theta + \sin(\theta+\alpha)}{\cos\theta - \cos(\theta+\alpha)}$ is independent of θ for $0 \leq \theta \leq \alpha$.
2. Can you find a geometric interpretation for this fact?