Thursday March 26, hours 32-33: Generalize

March-26-15 8:39 AM

Quiz; Quiz Sol'n; Then handout.

Additional Examples

~~1.4.2,~~ 2.2.6, ~~2.2.7,~~ 4.1.4, 5.1.3, 5.1.4, 5.1.9, 5.1.11, 5.4.4, 5.4.5, ~~5.4.6,~~ ~~5.4.7,~~ ~~6.9.2,~~ 7.4.4. Also, see Section 2.4 (Induction and Generalization).

2.2.6. If $a,b,c \geqslant 1$, prove that $4(abc+1) \geqslant (1+a)(1+b)(1+c)$. (Hint: Prove, more generally, that $2^{n-1}(a_1a_2\cdots a_n+1) \geqslant (1+a_1)(1+a_2)\cdots(1+a_n)$.)

4.1.4. Prove that there are no prime numbers in the infinite sequence of integers

$$10001, 100010001, 1000100010001, \ldots .$$

5.1.4. Show that

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

5.1.9. Sum each of the following:

(a) $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$.

(b) $1\times 2\binom{n}{2} + 2\times 3\binom{n}{3} + \cdots + (n-1)n\binom{n}{n}$.

(c) $\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}$.

(d) $\binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} - \cdots + (-1)^{n+1}n^2\binom{n}{n}$.

(e) $\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + (-1)^n\frac{1}{n+1}\binom{n}{n}$.

(f) $\sum_{j\geq 1}\left[(-1)^j\binom{n}{j-1}\Big/\sum_{1\leq k\leq j} k\right]$.

5.1.11. Prove the following identities:

(a) $$\frac{\binom{n}{1}}{1\times 2} - \frac{\binom{n}{2}}{2\times 3} + \frac{\binom{n}{3}}{3\times 4} - \cdots + (-1)^{n+1}\frac{\binom{n}{n}}{n(n+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1},$$

(b) $$\frac{\binom{n}{0}}{1^2} - \frac{\binom{n}{1}}{2^2} + \frac{\binom{n}{2}}{3^2} - \cdots + (-1)^n\frac{\binom{n}{n}}{(n+1)^2}$$

$$= \frac{1}{n+1}\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right].$$

5.4.4. Sum the infinite series

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots.$$

5.4.5. Sum the infinite series

$$1 - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{10} + \cdots.$$

(ends up involving an ugly integral)

7.4.4. On $[0, 1]$, let f have a continuous derivative satisfying $0 < f'(t) \leqslant 1$. Also, suppose that $f(0) = 0$. Prove that

$$\left[\int_0^1 f(t)\,dt\right]^2 \geqslant \int_0^1 [f(t)]^3\,dt.$$

2.4.1. If $A_1 + \cdots + A_n = \pi$, $0 < A_i \leqslant \pi$, $i = 1, \ldots, n$, then

$$\sin A_1 + \cdots + \sin A_n \leqslant n \sin\frac{\pi}{n}.$$

2.4.3. Let F_i denote the ith term in the Fibonacci sequence. Prove that $F_{n+1}^2 + F_n^2 = F_{2n+1}$.

2.4.5. Let S denote an n-by-n lattice square, $n \geqslant 3$. Show that it is possible to draw a polygonal path consisting of $2n-2$ segments which will pass through all of the n^2 lattice points of S.