

Thursday Feb 5, hours 14-15: Choose an Effective Notation, Exploit Symmetry

February-05-15 8:23 AM

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Handout for February 5, "Choose an Effective Notation" and "Exploit Symmetry"

Reading. Sections 1.5 and 1.6 of Larson's textbook.

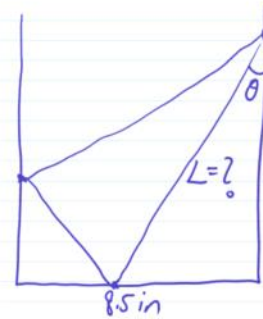
Next Quiz. Thursday February 12, on these two sections.

✓ **Problem 1** (Larson's 1.5.7). The bottom of this handout (width 8.5in) is folded up and to the left, so that the bottom right corner would align with the left edge of this page. Given the angle of the fold θ , what is the length of the crease L ? See the figure on the right.

✓ **Problem 2** (Larson's 1.5.8). Let P_1, P_2, \dots, P_{12} be the vertices of a regular 12-gon. Are the diagonals P_1P_9, P_2P_{11} and P_4P_{12} concurrent?

Problem 3 (Larson's 1.5.2).

1. If $n \in \mathbb{N}$ and $2n + 1$ is a square, show that $n + 1$ is the sum of two successive squares.
2. If $n \in \mathbb{N}$ and $3n + 1$ is a square, show that $n + 1$ is the sum of three successive squares.



Dror's Favourite "Choose an Effective Notation" problem is more a computer problem than a math problem, yet it is pretty and it will make us look at some pretty (and significant!) mathematics.

The *Kauffman Bracket* $\langle D \rangle$ of a knot diagram D is defined by the following recursive relations:

$$\langle \underbrace{\bigcirc \dots \bigcirc}_k \rangle = (-A^2 - A^{-2})^k, \quad \langle \begin{array}{c} \diagup \\ \diagdown \end{array} \rangle = A \langle \begin{array}{c} \diagdown \\ \diagup \end{array} \rangle + A^{-1} \langle \begin{array}{c} \diagup \\ \diagdown \end{array} \rangle$$

(where A is some variable).

Remark. The celebrated "Jones Polynomial" of D is very easily defined from $\langle D \rangle$,

$$J(D) = (-A^3)^{w(D)} \frac{\langle D \rangle}{-A^2 - A^{-2}} \Big|_{A \rightarrow q^{1/4}}$$

where $w(D)$ is the *writhe* of D , the difference between the number of right-handed and left-handed crossings in D . Yet this matters not for us today.

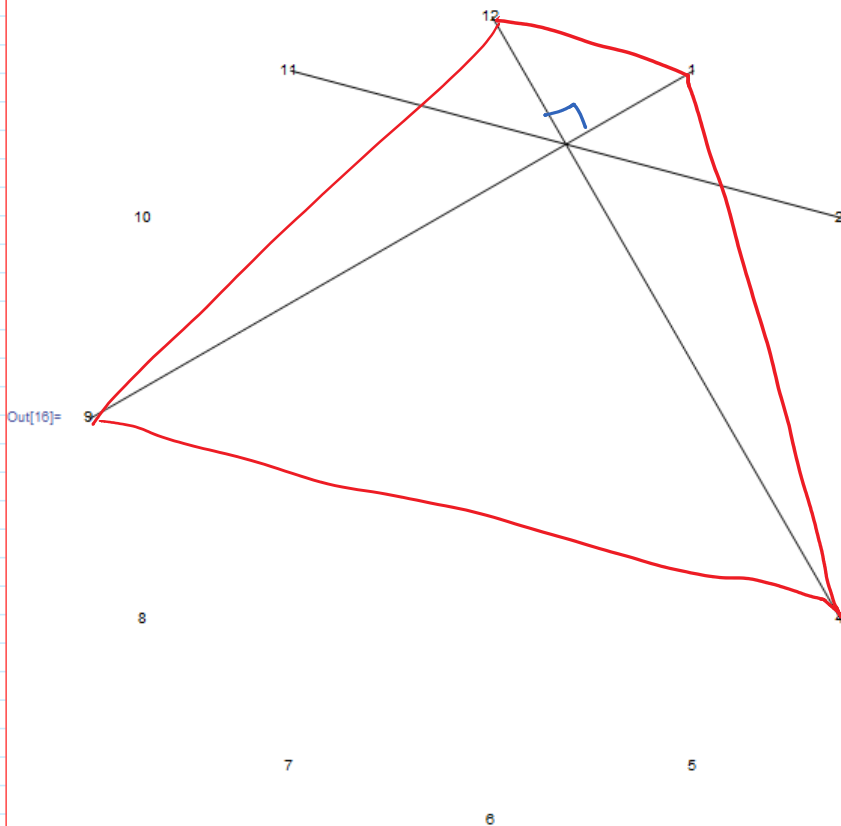
Homework Problem (not to be submitted, yet fame for an excellent solution). Write a very short computer program that will be able to compute the Kauffman bracket (and hence the Jones polynomial) of the following three knots:

<http://www.math.toronto.edu/~drorbn/classes/15-475-ProblemSolving/Feb05.html>

Problem 2.

```
In[15]= P_k := {Cos[2 π  $\frac{3-k}{12}$ ], Sin[2 π  $\frac{3-k}{12}$ ]};
```

```
Graphics[{
  Table[Text[k, P_k], {k, 12}],
  Line[{P_1, P_9}], Line[{P_2, P_11}], Line[{P_4, P_12}]
}]
```



$$\operatorname{Re}(\bar{a} z) = b \quad i(z_0 - z_1)$$

$$\operatorname{Im}(\bar{z}_1 - \bar{z}_0) z = \operatorname{Im} \bar{z}_1 z_0$$

$$\operatorname{Im}(\overline{e_1 - e_9}) \frac{e_2 + e_{11}}{2} \stackrel{?}{=} \operatorname{Im} \bar{e}_1 e_9$$

```
In[19]= e_k := e^{2π i k/12};
```

```
{Im[Conjugate[e_1 - e_9]  $\frac{e_2 + e_{11}}{2}$ ], Im[Conjugate[e_1] e_9]} // Simplify
```

```
Out[20]= { - $\frac{\sqrt{3}}{2}$ , - $\frac{\sqrt{3}}{2}$  }
```

Problem 3

$$1. \quad 2n+1 = (2k+1)^2 \Rightarrow 2n = 4k^2 + 4k$$

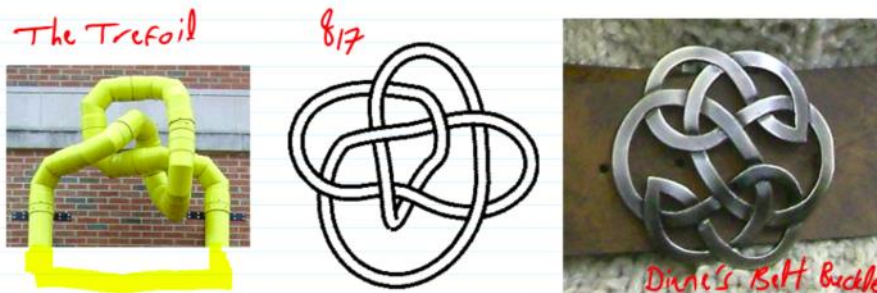
$$n = 2k^2 + 2k \quad n+1 = k^2 + (k+1)^2$$

$$2. \quad 3n+1 = (3k+1)^2 \quad \text{or} \quad (3k-1)^2$$

Case I

$$3n = 9k^2 + 6k \quad n+1 = 3k^2 + 2k + 1$$

$$= k^2 + k^2 + (k+1)^2$$



Problem 4 (Larson's 1.6.2).

1. Of all the rectangles which can be inscribed in a given circle, which has the greatest area?
2. Maximize $\sin \alpha + \sin \beta + \sin \gamma$, where α, β, γ are the angles of a triangle.
3. Of all the triangles of a fixed perimeter, which has the greatest area?
4. Of all the parallelepipeds of volume 1, which has the smallest surface area?
5. Of all the n -gons which can be inscribed in a given circle, which has the greatest area?
6. Dror adds: of all the n -gons with a given perimeter, which has the greatest area?

Problem 5 (Larson's 1.6.4). Let P be a point on the graph G of $y = f(x)$, where f is a cubic polynomial. Assume the tangent to the curve at P intersects G again at a point Q . Let A be the area bound by G and the segment PQ , and let B be the area defined in exactly the same manner, except starting with Q rather than with P . What is the relationship between A and B ?

Problem 6. A function on the plane has the property that the sum of its values on the corners of any square (of any size and orientation) is zero. Prove that the function is the zero function.

Problem 7. Prove: You cannot colour the points of the plane with just three colours, so that no two points of distance 1 will be coloured with the same colour. What if you had four colours available?

Dror's Favourite "Exploit Symmetry" problem: Two players alternate placing 1×1 , 2×1 , and 4×1 lego pieces on a circular table of diameter 100, with no overlaps. If a player cannot place a further piece, (s)he loses. Would you rather be the first to move or the second?

Dror's Favourite "Explore Symmetry" topic: See <http://drorbn.net/Treehouse>:

Video, handout, links at drorbn.net/Treehouse

The Basic Features.

3

rotation only

3

rotation-reflection

M

free mirror-reflection

G

free glide-reflection

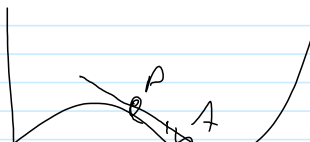
Gotta catch 'em all!

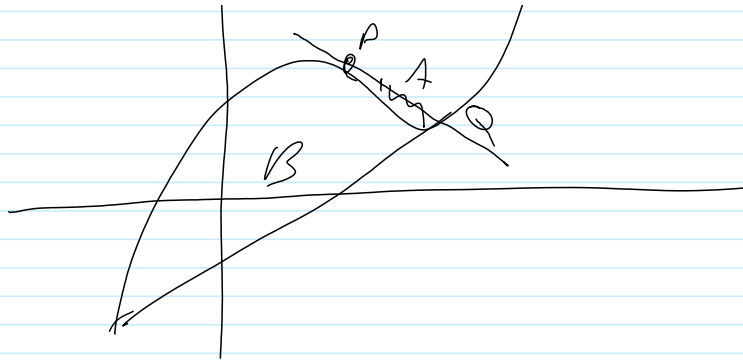
Theorem. There are precisely 17 patterns with which to tile the plane, no more, no less. They are all made of combinations of the 10 basic features, 2, 3, 4, 6, 2, 3, 4, 6, M, and G, as follows:

✓	Dror's	Conway's	crystallo-graphic	✓	Dror's	Conway's	crystallo-graphic
	2222	2222	p2		33	3*3	p31m
	333	333	p3		222	2*22	cmm
	442	442	p4		22M	22*	pmg
	632	632	p6		MM	**	pm
	2222	*2222	pmm		MG	*o	cm
	333	*333	p3m1		GG	oo	pg
	442	*442	p4m		22G	22o	pgg
	632	*632	p6m		∅	0	p1
	42	4*2	p4g				

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problem 5.





WLOG, $y = x^3 + ax$ $P = (x_0, y_0)$

$Q = (-2x_0, \dots)$

$\frac{y - y_0}{x - x_0} = 3x_0^2 + a$

$A = \int_{-2x_0}^{x_0} (x^3 + ax - \dots) dx$

$y = (3x_0^2 + a)(x - x_0) + y_0$

In[2]:= $\int_{-2x_0}^{x_0} (at + t^3 - ax_0 - x_0^3 - (t - x_0)(a + 3x_0^2)) dt$

Out[2]= $\frac{27x_0^4}{4}$