UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

APRIL EXAMINATIONS 2015

Math 475H1 Problem Solving Seminar — Final Exam

April 21, 2015

Solve 8 of the following 11 problems. Indicate clearly which problems you wish graded; otherwise an arbitrary subset of the problems you have attempted will be chosen for marking. The problems carry equal weight.

Duration. You have 3 hours to write this exam.

Allowed Material. Stationary only.

Good Luck!

Problem 1 (Larson's 1.1.4). Find positive natural numbers n and $a_1, a_2, \ldots, a_n$ such that the~~ir~~ sum $a_1 + \cdots + a_n$ is 1000 and the~~ir~~ product $a_1 a_2 \cdots a_n$ is as large as possible.

Problem 2 (Larson's 1.2.6). Let ABC be an acute-angled triangle (all angles below 90°), and let D be on the interior of the segement AB. Describe how one can find points E on AC and F on BC such that the triangle DEF will have the minimal possible perimeter.

Problem 3 (Larson's 1.3.1). Let n be a natural number. Find a general formula for the nth derivative of $f(x) = 1/(1 - x^2)$.

Problem 4 (Larson's 1.5.4). Let $-1 < a_0 < 1$ and define recursively for $n > 0$,

$$a_n = \left(\frac{1 + a_{n-1}}{2}\right)^{1/2}.$$

What happens to $4^n(1 - a_n)$ as $n \to \infty$?

Problem 5 (Larson's 1.6.2e). Let n be a natural number. Of all the n-gons which can be inscribed in a given circle, which has the greatest area?

Problem 6 (Larson's 1.7.8). Determine $F(x)$, if for all real x and y, $F(x)F(y) - F(xy) = x + y$.

Problem 7 (Larson's 2.6.1). Let n be a natural number. Given $n+1$ positive integers, none of which exceeds $2n$, show that one of these integers divides another of these integers.

Problem 8 (Larson's 5.4.1). Prove that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ is an irrational number.

Problem 9 (Larson's 1.10.8). Let n be odd and σ a permutation of $\{1, 2, \ldots, n\}$. Prove that the product

$$(\sigma_1 - 1)(\sigma_2 - 2) \cdots (\sigma_n - n)$$

is even.

Problem 10 (Larson's 3.3.28). Prove that there are infinitely many primes of the form $6n - 1$.
Hint. Consider $(p_1 p_2 \cdots p_k)^2 - 2$.

Problem 11 (Larson's 1.12.4c, modified) Compute the sum $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k}$.

Good Luck!

In red: post-exam markup.