Today. matrices & matrix algebra.

Riddle along: A game: Player A writes the numbers 1-18 on the faces of three blank dice, to her liking. Player B takes one of the 3 dice. Player B takes one of the remaining two, and throws away the third. Player A and B then play 1,000 rounds of "dice war" with the dice they hold. Whom would you rather be, player A or player B?

Reminders.

V/F, basis $\beta = (v_1 \dots v_n)$ W/F, basis $\gamma = (w_1 \dots w_m)$

Abstract, general, coord-free

merely numbers, choice-dependent, easy to work with

$$\mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$$

$$T \longrightarrow [T]_\beta^\gamma = A$$

$$A = \begin{pmatrix} a_{11} & & & a_{1n} \\ [Tv_1]_\gamma & [Tv_2]_\gamma & \dots & [Tv_n]_\gamma \\ a_{m1} & & & a_{mn} \end{pmatrix} \Longleftrightarrow Tv_j = \sum_{j=1}^{m} a_{kj} w_k$$

Examples. $0 \to (0)$, $I \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Examples

2. $D : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ differentiation
3. $T_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
4. $A : F^n \longrightarrow F^m$

Complete the proof that this is a vector space iso.

$$\begin{tikzcd} U \arrow[r, "S"', "B"] \arrow[rr, bend right=50, "T\circ S", "C"'] & V \arrow[r, "T"', "A"] & W \end{tikzcd}$$

basis: α, β, γ
dim: p, n, m

If you know A & B, Can you derive C?

Derive C, then...

Definition $A \in M_{m\times n}$, $B \in M_{n\times p}$ $A\cdot B \in M_{m\times p}$ by $(A\cdot B)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$

Example 1 $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \dots$

Thm $[T\circ S]_\alpha^\gamma = [T]_\beta^\gamma \cdot [S]_\alpha^\beta$

Example 2 $A \in M_{m\times n}$ $v \in M_{n\times 1} = F^n$

$Av \in M_{m\times 1} = F^m$ ......

what we called $T_A(v)$ is really Av.

| really Av.

Example 3 $T_\beta \circ T_\alpha = T_{\alpha+\beta}$ for rotations.

The good and the bad about "matrix algebra":

Good	Bad
1 $A+B=B+A$, $(A+B)+C=A+(B+C)$ (basically, all works for addition)	1. Addition is defined only for matrices of same dims.
2. $A(B\cdot C)=(A\cdot B)C$ $\exists I$ s.t. $A\cdot I=A$, $IA=A$	2. mult. is defined only if "mid" dimension matches, & produces an output of yet other dims.
3 If $A\cdot A^{-1}=I$, then $A^{-1}A=I$	3. A^{-1} may not exist even if $A\neq 0$.
4. $(A+B)C=AC+BC$ $A(B+C)=AB+AC$	4. Generally, $A\cdot B\neq B\cdot A$, even when both make sense.

done line

target line

Next goals: 1. Compute rank T/A. 2. Compute A^{-1} (when possible)
3. Solve systems of linear eqn's.

Proposition Given $V' \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W'$ with invertible P&Q, rank $T=$ rank PTQ [enough that Q surjective & P injective]

Pf

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "P"] \\ V' \arrow[u, "Q"] \arrow[r, "T'=", "PTQ"'] & W' \end{tikzcd}$$

$W \supset im(T)=C$, basis $=(w_i=T(v_i))_{i=1}^{r}$

$\Downarrow$

$W' \supset im(T')=C'$ basis $=(w_i'=P(w_i))_{i=1}^{r}$

Need: 1. $w_i' \in im\, T'$; meaning $\exists v_i' \in V'$ s.t. $w_i'=T'v_i'$

2. w_i' span C'

3. w_i' are lin. indep.

Def If $A \in M_{m\times n}$, let rank $A :=$ rank T_A, where T_A is the "standard" $T_A: F^n \to F^m$

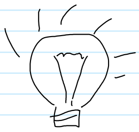
Comment 1 rank $[T]_\beta^\gamma =$ rank T Pf.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\,]_\beta}"'] & W \arrow[d, "{[\,]_\gamma}"] \\ F^n \arrow[r, "A", "T_A"'] & F_m \end{tikzcd}$$

Comment 2 rank $A=$ rank PAQ whenever

$T \quad T_A \quad m$

Comment 2 rank A = rank PAQ whenever $P \in M_{m\times m}$ & $Q \in M_{n\times n}$ are invertible.



Look for P & Q that will make PAQ "simpler" than A.