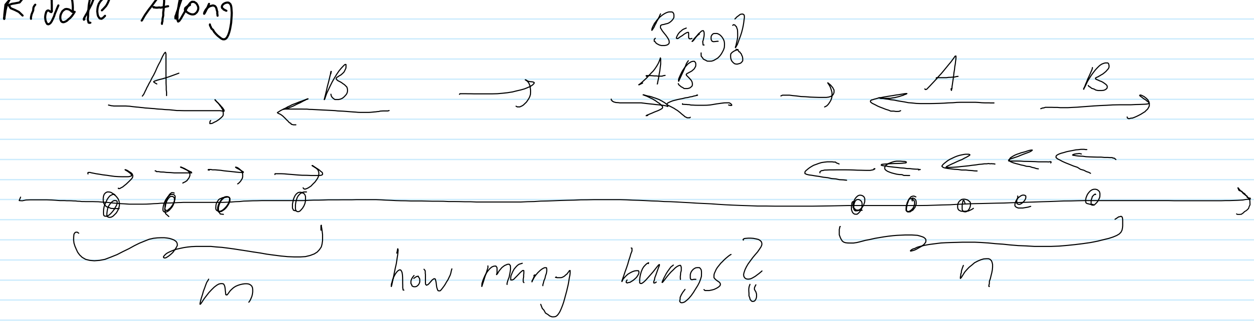
240Algebral-141022 Hours 19-20: Lagrange interpolation, linear tansformations

October-21-14 1:36 PM

Read Along 1.6, 2.1 HW5 on web & print.

Riddle Along



Today: Lagrange interpolation, linear transformations.

Reminder $x_1, \ldots x_{n+1} \in F$ distinct $y_1, \ldots y_{n+1} \in F$ any

$\exists!\ p \in P_n$ s.t. $p(x_i) = y_i$?

$$\tilde{p}_i(x) = \prod_{j \neq i} (x - x_j) \qquad \tilde{p}_i(x_j) = \begin{cases} 0 & j \neq i \\ \neq 0 & i = j \end{cases}$$

Example: $x_{1,2,3} = 0, 1, 3 \quad y_{1,2,3} = 5, 2, 2$

$p(0) = 5 \quad p(1) = 2 \quad p(3) = 2$

$\tilde{p}_1 = (x - x_2)(x - x_3) = (x-1)(x-3) = x^2 - 4x + 3$

$\tilde{p}_2 = (x - x_1)(x - x_3) = x^2 - 3x$

$\tilde{p}_3 = (x - x_1)(x - x_2) = x^2 - x$

on Fresh BB

$\ldots\ldots \quad p = x^2 - 4x + 5$

start line

set $p_i(x) = \tilde{p}_i(x) / \tilde{p}_i(x_i) = \ldots$

Then * $p(x) := \sum y_i p_i(x)$ satisfies $p(x_i) = y_i$

* $\beta = \{p_1, \ldots, p_{n+1}\}$ is lin. indep.
* $\Rightarrow \beta$ is a basis
* Every $f \in P_n(\mathbb{R})$ can be expressed as a lin. comb. of the p_i in a unique way.
* If $q(x)$ also satisfies $q(x_i) = y_i$, then $q(x) = p(x)$.
* Therefore the solution to our problem is unique
* Aside: If $\forall i,\ p(x_i) = 0$, then $p = 0$

(so a non-zero polynomial of degree n has at most n roots.)

A word about "morphisms".

- "T:V->W is linear"
- Preserving 0.
- Claim on cx+y.
- Claim on differences and many-element sums. flipped.
- Example: R^2->R^2 by explicit formula.
- Example: Differentiation, multiplication by x.
- Example: Matrices and linear transformations on F^n.
- Example: Rotation (+ explicit formula).

done line

- Added 2012: \calL(V,W) is a vector space.
- Composition of linear trans is a linear trans.
- Composition is non-commutative. Example: differentiation and multiplication by x.

target line

- A linear transformation is determined by its values on a basis, and these are arbitrary.
- "Isomorphism".

Def V & W are isomorphic if $\exists$ l.t. $T: V\to W$ and $S: W\to V$ s.t. $S\circ T = I_V$ & $T\circ S = I_W$

Thm If V, W are f.d. over F, then $\dim V = \dim W$ iff V is isomorphic to W.

Corollary If $\dim V = n$ over F, V is isomorphic to F^n.

Two "mathematical structures" are "isomorphic" if there's a bijection (1-1 & onto corres.) between their elements which preserves all relevant relations.

Example plastic chess is iso. to ivory chess, but not to checkers.

Example The game of 15.

Pf of Thm & of corollary

Fix a l.t. $T: V\to W$

Def $N(T) = \ker T = \{v : Tv = 0\}$ "null space", "kernel".

$R(T) = \operatorname{im} T = \{Tv : v\in V\}$ "range", "image"

Prop/Def $N(T)\subset V$ is a subspace; $\text{nullity}(T) := \dim N(T)$

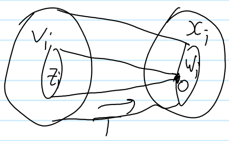
$R(T)\subset W$ is a subspace; $\text{rank}(T) := \dim R(T)$

Examples 0, I_V, $D: P_n(\mathbb{R})\to P_n(\mathbb{R})$

Thm "the dimension theorem", "the rank-nullity Thm"

Given $T: V\to W$, $\dim V = \text{rank}(T) + \text{nullity}(T)$

Pf $(z_i)_1^n$ basis of $N(T)$, extend to $(z_i) \cup (v_i)$ a basis of V,



claim $w_i := T(v_i)$ are lin indep. in W Pf ....

claim w_i span $R(T)$ pf . . .