240AlgebraI-141015 Hours 16-17: Bases and replacement

September-18-14 6:29 PM

Read Along sections 1.5–1.7.

20-minutes on symmetries: Fri 6:30PM, drorbn.net/Treehouse

Riddle Along A mirror swaps left and right, but not up and down. How on Earth does the mirror know??

HW4 is due tomorrow!

Term test on Tue Oct 21 1:10–3PM, HS 610.

My expectations: 1. complete Mastery of the material.
2. Yes, meaning every single proof.

Find old TT's on previous year's web sites

TA office hours – about 10 hours on Fri, Mon, Tue; details on web by midnight.

Today: bases. (our first <u>real</u> thm)

Reminder. $\beta=(u_1,\dots u_n)$ is a basis iff every $u\in V$ can be written in a unique way as a l.c. $u=\sum\alpha_i u_i$.

<u>Thm</u> If a finite set S generates a v.s. V, then there is a subset $\beta\subset S$ which is a basis of V.

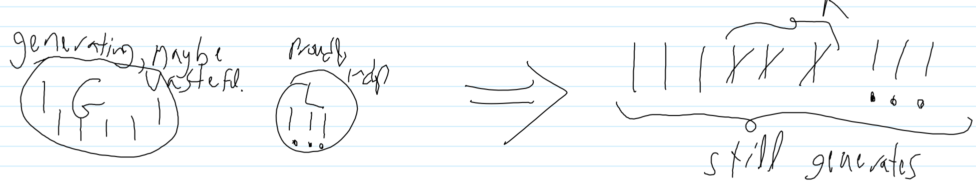
<u>Thm</u> If a v.s. V has a finite basis, then every other basis of V has the same number of elements in it.
(allows "$\dim V$")

<u>Lemma</u> (the replacement lemma) $\operatorname{span} G=V$, L lin indep w/ $|L|=n$

$\Longrightarrow \exists R\subset G$ with $|R|=n$ and $\operatorname{span}((G\setminus R)\cup L)=V$

in particular, $|L|\leq|G|$

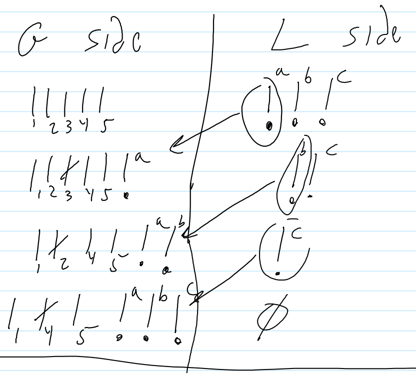
Post Mortem: I should have left out the "in particular" and instead I should have phrase two corollaries to the lemma:
1. $|L|\leq|G|$
2. If a generating G is finite, then every lin. indep. L is finite too.



Pf of theorem from lemma.

Informal proof of Lemma First of all, if $\sum a_i u_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

Now:



Formal proof: Induction on $|L|$. $|L|=0$: trivial.

Now $|L| = n+1$; $L = \{v_1 \ldots v_{n+1}\}$. Use $L' = \{v_1 \ldots v_n\}$, Find $R' = \{r_1 \ldots r_n\} \subset G$ s.t. $(G \setminus \{r_1 \ldots r_n\}) \cup \{v_1 \ldots v_n\}$ spans. write

$$v_{n+1} = \sum_1^m a_i u_i + b_1 v_1 + \ldots + b_n v_n \qquad u_i \in G \setminus \{r_1 \ldots r_n\}$$

$\therefore$ Not all $a_i = 0$; let j be such that $a_j \neq 0$

$\therefore$ so $u_j \in \operatorname{span}(u_1 \, \cancel{u_j} \, u_m \, v_1 \ldots v_{m+1})$,

So take $r_{n+1} = u_j$ & $R = R' \cup \{r_{n+1}\}$.

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$. $\Longrightarrow$ "$\dim V$" makes sense.

2. Assume $\dim V = n$. Then

a. If G generates V, $|G| \geq n$ & if also $|G| = n$, then G is a basis.

b. If L is linearly indep in V, the $|L| \leq n$; if also $|L| = n$, L is a basis. if also $|L| < n$, L can be extended to a basis.

3. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \leq \dim V$.

done line

If also $\dim W = \dim V$, then $W = V$.

If also $\dim W < \dim V$, then any basis of W
can be extended to a basis of V.

Fishy Thm: Every v.s. has a basis. [Including $\mathbb{R}/\mathbb{Q}$]

The Lagrange interpolation formula:

Let x_i be distinct pts in $\mathbb{R}/\mathbb{F}$ $\quad i=1,\dots,n+1$

Let y_i be any pts in $\mathbb{R}/\mathbb{F}$.

Q Can you find a polynomial $p \in P_n(\mathbb{R})$ s.t. $p(x_i)=y_i$?

Is it unique?

Who cares? * Scientists.

* Computer drawing programs.

Follow through w/ example.

$p(0)=5$
$p(1)=2$
$p(3)=2$

$p_1 = \frac{(x-1)(x-3)}{3} = \frac{1}{3}(x^2-4x+3)$

$p_2 = \frac{x(x-3)}{-2} =$

$p_3 = \frac{x(x-1)}{6} = \dots$

$p = x^2-4x+5$

Solution

Let $\tilde{p}_i(x) = \prod_{j\neq i}(x-x_j)$

Then $p_i(x_j) = \begin{cases} 0 & j\neq i \\ \neq 0 & i=j \end{cases}$

set $p_i(x) = \tilde{p}_i(x)/\tilde{p}_i(x_i) = \dots$

Then * $p(x) := \sum y_i p_i(x)$ satisfies $p(x_i)=y_i$

* $\beta = \{p_1 \dots p_{n+1}\}$ is lin. indep.
* $\Rightarrow \beta$ is a basis
* Every $f \in P_n(\mathbb{R})$ can be expressed as a lin. comb. of the p_i in a unique way.
* If $q(x)$ also satisfies $q(x_i)=y_i$, then $q(x)=p(x)$.
* Therefore the solution to our problem is unique
* Aside: If $\forall i$, $p(x_i)=0$, then $p=0$

(so a non-zero polynomial of degree n has at most n roots.)