Name: ________________________________ Student ID: ___________________

UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2014

Math 240H1 Algebra I — Makeup Final Exam

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Solve 4 of the following 5 questions. If you solve more than 4 questions indicate very clearly below which are the ones that you want marked, or else a random one will be left out. Please write your answers within this booklet. You may also use the back side of the pages for answers and you may continue answers in the blank pages provided at the end of this notebook (though indicate "continued on page ..." when you do so). You may mark pages "draft" and use them for scratch work.

The questions carry equal weight though different parts of the same question may be weighted differently.

There are 3 pages in this exam booklet.

Duration. You have 3 hours to write this exam.

Allowed Material. Stationary and basic calculators, not capable of displaying text or sounding speech. A stuffed animal for personal comfort.

Good Luck!

For grading use — circle the 4 questions that you want marked:

1	/25	4	/25
2	/25	5	/25
3	/25		
Total	/100		

Problem 1. Let A be a matrix in $M_{2\times 2}(F)$. Let $T\colon M_{2\times 2}(F) \to M_{2\times 2}(F)$ be defined by $T(B) = A^T BA$.

1. Show that T is a linear transformation.
2. Find the matrix representing T relative to the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

   which is taken as a basis of both the domain space of T and the target space of T.

Tip. In math-talk, "show" means "prove".
Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.
Tip. Neatness, cleanliness and organization count, here and everywhere else!
Problem 2. State and prove the "dimension theorem", also known as the "rank-nullity theorem", for a given linear transformation $T\colon V \to W$.

Tip. In math-talk, "state" means "write the statement of, in full".
Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.
Problem 3. Find all the solutions (if any exist) of the following two systems of linear equations:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Tip. Show all intermediate steps!
Tip. It is always an excellent idea to substitute your solutions back into the equations and see if they really work.
Problem 4. Let A be the matrix $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$.

1. Compute $\det(A - \lambda I)$.
2. Find the eigenvalues λ_1 and λ_2 of A.
3. Find their corresponding eigenvectors v_1 and v_2.
4. Find a matrix C for which $AC = CD$, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
5. Compute the inverse of C.
6. For an arbitrary natural number n, compute A^n by computing CD^nC^{-1}.

Problem 5. In the spaces provided, indicate for each of the following statements if it is TRUE or FALSE. You may leave an entry blank. You will earn 2 points for each correct answer, (-3) points for each incorrect answer, and 0 points for each entry left blank. Your overall mark for this question will be raised to 0 if you earn a negative number of points.

If nothing is otherwise stated, A and B are assumed to be square $n \times n$ matrices with entries in some field F.

1. ____________ If a row of A is identical to one of the columns of A, then $\det(A) = 0$.

2. ____________ The determinant of A can be evaluated using an expansion along any row or column.

3. ____________ If two rows of A are identical then $\det(A) = 0$.

4. ____________ If B is obtained from A by interchanging any two columns, then $\det(B) = -\det(A)$.

5. ____________ If B is obtained from a A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

6. ____________ The function $\det\colon M_{n\times n}(F) \to F$ is a linear transformation.

7. ____________ If B is obtained from A by adding c times one row to another row, then $\det(B) = c\det(A)$.

8. ____________ If A is an $n \times n$ matrix where n is even, then $\det(A) = \det(-A)$.

9. ____________ If A has rank n, then $\det(A) \neq 0$.

10. ____________ If A is upper triangular (meaning that $A_{ij} = 0$ whenever $i > j$), then $\det(A)$ is equal to the product of the diagonal entries A_{ii} of A.

11. ____________ Always, $\det(A) = \det(A^T)$.

12. ____________ If $AB \neq BA$ then $\det(AB) \neq \det(BA)$.

13. ____________ If A is invertible, then $\det(A) = \det(A^{-1})$.

14. ____________ Always, $\det(AB) = \det(BA)$.

Tip. There is no need to justify your answers.
Tip. Note that a statement is considered TRUE only if it is **always** true, and not just **sometimes**.

Good Luck!