Regular singular points scratch

November-19-12
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The key: Find y_r s.t. $Ly_r = F(r)x^r$

The coefficients of y_r involve inverting $F(r+n)$ for every $n \geq 1$; yet each one gets inverted <u>just</u> <u>once</u>.

Let the roots of $F(r)=0$ be r_1 & r_2.

If $r_1 - r_2 \notin \mathbb{Z}$, y_{r_1} & y_{r_2} solve $Ly=0$

If $r_1 = r_2$, the solutions of $Ly=0$ are

$$y_{r_1} \quad \& \quad \frac{\partial}{\partial r} y_r \Big|_{r=r_1}$$

If $r_1 - r_2 = n \in \mathbb{N}_{>0}$, the solutions of $Ly=0$ are

$$y_{r_1} \quad \& \quad \frac{\partial}{\partial r}\left[(r-r_2)\, y_r\right] \Big|_{r=r_2}$$

Question. Is it always possible, by some logarithmic change of variables, to turn an RSP to an OP?

Question. In RSP problems, one solution is easy to get. Why not get the other by reduction of order?

$y'' + py' + qy = 0 \qquad y = uv, \quad u$ solves

$$\Rightarrow 2u'v' + uv'' + puv' = 0 \qquad \phi = v'$$

$$u\phi' + (2u' + pu)\phi = 0$$

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad V_1 =$$

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = \begin{pmatrix} 0 & x^{-1} \\ x^{-1}q & x^{-1}p \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad \begin{aligned} V_1' &= V_2/x \\ V_2' &= x^{-1}qV_1 + x^{-1}pV_2 \end{aligned}$$

$$V_1'' = \left(\frac{V_2}{x}\right)' = \frac{V_2' x - V_2}{x^2} = \frac{qV_1 + (p-1)V_2}{x^2}$$

$$x^2 V_1'' = qV_1 + (p-1)xV_1' \qquad \text{RSP!}$$

$$x^2 y'' + xpy' + qy = 0 \qquad x = e^t$$

$$\log x = t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} = e^{-t}\frac{dy}{dt} \qquad \frac{dt}{dx} = \frac{1}{x} = e^{-t}$$

$$\frac{dy^2}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \left(-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\left(\frac{1}{x}\frac{d^2y}{dt^2}\right)\right)$$

new line $$= \frac{dt}{dx}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right) = e^{-t}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)$$

$$= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

So eqn becomes

$$\ddot{y} - \dot{y} + p\dot{y} + qy = 0$$

$$\ddot{y} + (p-1)\dot{y} + qy = 0$$

$$P(x) = \sum a_n x^n = \sum a_n e^{nt}$$

$$xy' + py = 0 \qquad p = \sum p_k x^k \quad y = \sum a_k x^{\alpha+k}$$
$$a_0 = 1.$$

$$0 = (\alpha+k)a_k + \sum_{j=0}^{k} p_j a_{k-j} =$$

$$(\alpha+k+p_0)a_k + \sum_{j=1}^{k} p_j a_{k-j}$$

α must be $-p_0$

$$t\dot{x} = ax + by \qquad \frac{a}{t} \quad \frac{b}{t}$$
$$\dot{y} = cx + dy \qquad c \quad d$$

$$\lambda^2 - \left(\frac{a}{t} + d\right) + \frac{1}{t}(ad - bc) = 0$$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \to \lambda_{1,2} = 0, a \quad \begin{pmatrix} -b \\ a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a^2 & ab \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
$$a \quad 0$$
$$0 \quad \sim$$

$$\phi = x + \frac{b}{a}y \qquad \psi = \frac{1}{a}y$$
$$x = \phi - b\psi \qquad y = a\psi$$

$$t(\dot{\phi} - b\dot{\psi}) = a\phi - ab\psi + ba\psi = a\phi$$
$$a\dot{\psi} = c\phi - bc\psi + da\psi$$

$$\to \quad t\dot{\phi} = a\phi + \frac{tb}{a}(c\phi - bc\psi + ad\psi)$$

$xy'' + py' + q = 0$

A special case

$\alpha(\alpha-1) + p\alpha = 0$

$\alpha^2 - \alpha + p\alpha = 0$

$\alpha + (p-1) = 0 \qquad \alpha = 1-p$

$x^2 y'' + py' + q = 0 \qquad \sum a_n x^{n+\alpha}$

$(n+\alpha)(n+\alpha-1)a_n + a_{n+1}(n+\alpha+1)p + a_n$

a_{n+1} grows factorially. A "harder" singularity.