Fuchs' theorem scratch

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$$y'' + \frac{1}{1-x} y = 0 \qquad y = \sum a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} a_k \right] x^n = 0$$

$$a_0 = 1 \qquad a_3 = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12}$$

$$a_1 = 0 \qquad a_4 = \frac{1}{12} \ldots$$

$$a_2 = -\frac{1}{2}$$

$$y'' + p(x) y' + q(x) y = 0$$

$$y' = \sum n a_n x^{n-1} = \sum (n+1) a_{n+1} x^n$$

$$y'' = \sum (n+2)(n+1) a_{n+2} x^n$$

$$(n+2)(n+1) a_{n+2} + \sum_k p_{n-k} (k+1) a_{k+1} + \sum_k q_{n-k} a_k = 0$$

<u>Claim</u> If $|p_n|, |q_n| \leq C_1 \rho^n$ then $|a_n| \leq C_2 \rho^n$

<u>pf</u> By induction. No issue for a_0 & a_1. Then

$$a_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k \right] \qquad \text{So}$$

$$|a_{n+2}| \leq \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) |p_{n-k}| |a_{k+1}| + |q_{n-k}| |a_k| \right]$$

$$\leq \frac{C_1 C_2}{(n+1)(n+2)} \sum_{k=0}^{n} \left((k+1) \rho^{n+1} + \rho^n \right)$$

$$\leq \frac{\quad}{(n+1)(n+2)} \sum_{k=0} \left((k+1) \rho^{n+1} + \rho^{n} \right)$$

$$= C_1 C_2 \left(\frac{1}{2} \rho^{n+1} + \frac{1}{n+2} \rho^{n} \right) \leq$$

Fails!

$y' = 10y \qquad (n+1)a_n = 10 a_{n-1}$

$$a_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k \right]$$

<u>Claim</u> If $|p_n|, |q_n| \leq C_1 \rho_1^n$ & $\rho_2 > \rho_1$, then

for some C_2, $|a_n| \leq C_2 \rho_2^n$.

<u>Pf</u> By induction.

$$|a_{n+2}| \leq \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) |p_{n-k}| |a_{k+1}| + |q_{n-k}| |a_k| \right]$$

$$\leq \frac{C_1 C_2}{(n+1)(n+2)} \sum_{k=0}^{n} \rho_1^{n-k} \left((k+1) \rho_2^{k+1} + \rho_2^{k} \right)$$

$$= \frac{C_1 C_2}{(n+1)(n+2)} \sum_{k=0}^{n} \rho_2^{n-k} \left(\rho_1 / \rho_2 \right)^{n-k} \left((k+1) \rho_2^{k+1} + \rho_2^{k} \right) \Big)$$

$$= \frac{C_1 C_2}{(n+1)(n+2)} \rho_2^{n} \sum_{k=0}^{n} \left(\frac{\rho_1}{\rho_2} \right)^{k} \left((n-k+1) \rho_2 + 1 \right)$$

Want: $a_{n+1} \leq \left(\rho_2 + \frac{c}{n} \right) a_n$?