Discuss the Final.

Do the course evaluations!

4.2. **Changing the Independent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $z = \nu(x)$, where ν satisfies $\nu'' + p\nu' = 0$, then the equation becomes

$$\frac{d^2y}{dz^2} + Q(z)y = 0, \qquad \text{where} \qquad Q(z) = \frac{q(x(z))}{[\nu'(x(z))]^2}.$$

The zeros of y get moved by this transformation, so studying the oscillatory behaviour of $y(x)$ as $x \to \infty$ corresponds to studying the oscillatory behaviour of $y(z)$ as $z \to \lim_{x\to\infty} \nu(x)$, and the latter point may or may not be ∞. Note though, that the amplitudes of oscillations (if they occur), are unchanged.

on board.

Example 4.2. Under the change of independent variable $z(x) = x^3/3$, the equation $y'' - \frac{2}{x}y' + y = 0$ becomes the equation $\frac{d^2y}{dz^2} + \frac{1}{(3z)^{4/3}}y = 0$:

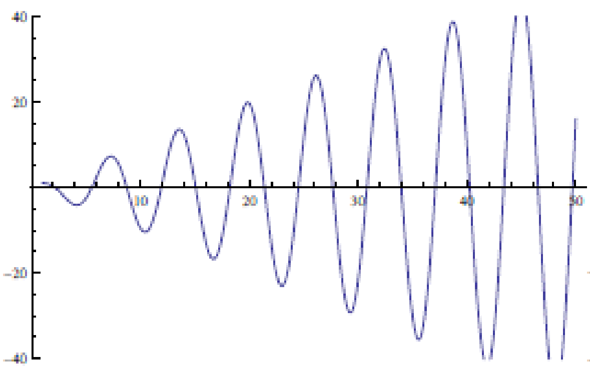
Indeed, $\nu'' - \frac{2}{x}\nu' = 0 \Rightarrow \nu' = x^2 \Rightarrow z = \nu = x^3/3$,

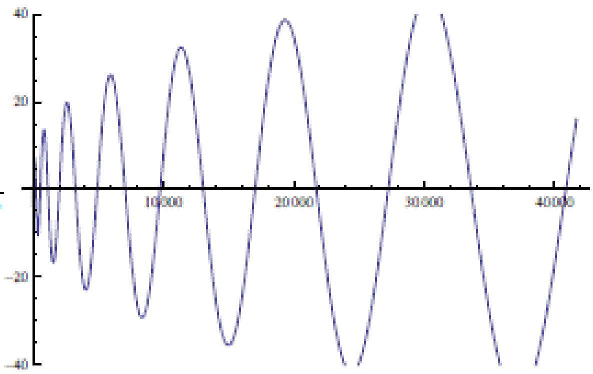
So $x = \sqrt[3]{3z}$ so $Q = \dfrac{1}{((\sqrt[3]{3z})^2)^2} = \dfrac{1}{(3z)^{4/3}}$

```
a = 50; b = 40;
ψ = NDSolve[
   y''[x] - 2/x y'[x] + y[x] == 0
    && y[1] == 1 && y'[1] == 0,
   y[x], {x, 1, a}
  ];
Plot[Evaluate[y[x] /. ψ],
 {x, 1, a}, PlotRange → {-b, b}]
```

```
ψ = NDSolve[
   y''[z] + 1/(3 z)^(4/3) y[z] == 0
    && y[1/3] == 1 && y'[1/3] == 0,
   y[z], {z, 1, a^3/3}
  ];
Plot[Evaluate[y[z] /. ψ],
 {z, 1, a^3/3}, PlotRange → {-b, b}]
```

Handout page 8





Theorem 6.1. *Consider a solution y of the equation $y'' + py' + qy = 0$. If $q > 0$ and $q' + 2pq > 0$ on some interval $[a, b]$ and $y'(a) = 0 = y'(b)$, then $|y(a)| > |y(b)|$. If instead $q' + 2pq < 0$ and $y'(a) = 0 = y'(b)$, then $|y(a)| < |y(b)|$. Similarly for non-strict inequalities.*

Proof. Consider $F = y^2 + \frac{(y')^2}{q}$ and note that $F' = -(q' + 2pq)\frac{(y')^2}{q^2}$. $\square$

Example 6.1. For Bessel's equation $y'' + \frac{1}{x}y' + (1 - \alpha^2/x^2)y = 0$ we have $q' + 2pq = 2/x > 0$, and hence the amplitudes of its oscillations decreases on $x > 0$. Yet for $y'' + y/x^2 = 0$ we have $q' + 2pq = \frac{-2}{x^3} < 0$, and hence the amplitudes of its oscillations increases on $x > 0$.

Theorem 6.1 has the following "opposite" (really, strengthening):

Proposition 6.2. *Under the same conditions as in the theorem, let P be some primitive of p, meaning $P' = p$. Then*

$$e^{P(a)}\sqrt{q(a)}|y(a)| < e^{P(b)}\sqrt{q(b)}|y(b)| \qquad \text{if} \qquad q' + 2pq > 0,$$

and

$$e^{P(a)}\sqrt{q(a)}|y(a)| > e^{P(b)}\sqrt{q(b)}|y(b)| \qquad \text{if} \qquad q' + 2pq < 0.$$

Proof. Use the auxiliary function $G(x) = e^{2P}(qy^2 + (y')^2)$. $\square$

Corollary 6.2. If $y'' + qy = 0$ where $q(x) \to L > 0$ monotonically as $x \to \infty$, then y oscillates as $x \to \infty$ with amplitudes that approach a finite, non-zero level.

Exercise 6.1. Describe, as best as you can at this stage, the behaviour as $x \to \infty$ of solutions of the equation $y'' + \left(1 - \frac{2}{x^2}\right) y = 0$. skipped.

Example 6.3. Under the transformation $v = \sqrt{x}y$ Bessel's equation $y'' + \frac{1}{x}y + \left(1 - \frac{\alpha^2}{x^2}\right) y = 0$ becomes the equation

$$v'' + \left(1 + \frac{1 - 4\alpha^2}{4x^2}\right) v = 0.$$

Thus we see that the oscillations of v increase if $\alpha < \frac{1}{2}$ and decrease if $\alpha > \frac{1}{2}$. Further, they approach a constant level — but this means that the oscillations of y decrease like $\frac{1}{\sqrt{x}}$.