Discussion of the Final: Tomorrow.

Theorem 5.1. *(The Sturm Comparison Theorem) Suppose y_1 satisfies $y_1'' + q_1 y_1 = 0$ and y_2 satisfies $y_2'' + q_2 y_2 = 0$ and suppose $q_2 > q_1$ in some interval. Then in the open interval between any two zeros of y_1 there is a zero of y_2 (hence y_2 oscillates more rapidly than y_1).*

} on board.

Corollary 5.1. Assuming $y'' + qy = 0$, if q is increasing the the distance between successive zeros of y is decreasing, and if q is decreasing then the distance between successive zeros of y is increasing.

Example 5.2. As we have seen in Example 4.1 the Bessel equation of order 0 is equivalent to the equation $V'' + (1 + \frac{1}{4x^2})V = 0$. Hence the distance between successive zeros of the Bessel equation of order 0 is increasing and by comparison with $v'' + v = 0$, it converges to π:

```
zs = x /. Table[FindRoot[y[x] /. J0, {x, λ}], {λ, 2.8, 50, 3.14}]
{2.91009, 6.03123, 9.16593, 12.3041, 15.4436, 18.5839, 21.7245, 24.8654,
 28.0064, 31.1475, 34.2888, 37.43, 40.5714, 43.7127, 46.8541, 49.9956}
Table[zs[[j + 1]] - zs[[j]], {j, 1, 15}]
{3.12114, 3.1347, 3.13816, 3.13954, 3.14023, 3.14062, 3.14087,
 3.14103, 3.14114, 3.14123, 3.14129, 3.14133, 3.14137, 3.1414, 3.14143}
```

Added Dec 18, 2012:
I could have used this as an example for the "change indep var then change dep var" sequence for getting ultra-slow oscillations, starting w/ $y'' + \gamma y = 0$

```
In[1]:= eq = y''[x] + γ y[x] /. y → (Y[e^#] &) /.
            x → Log[z];
        eq = Expand[eq / Coefficient[eq, Y''[z]]]
```

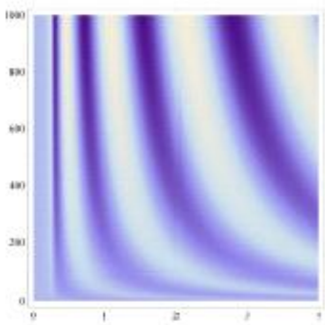
$$\text{Out[2]= } \frac{\gamma Y[z]}{z^2} + \frac{Y'[z]}{z} + Y''[z]$$

```
In[3]:= {p, q} = Coefficient[eq, #] & /@
            {Y'[z], Y[z]};
        Q = q - 1/4 p^2 - 1/2 ∂z p // Simplify
```

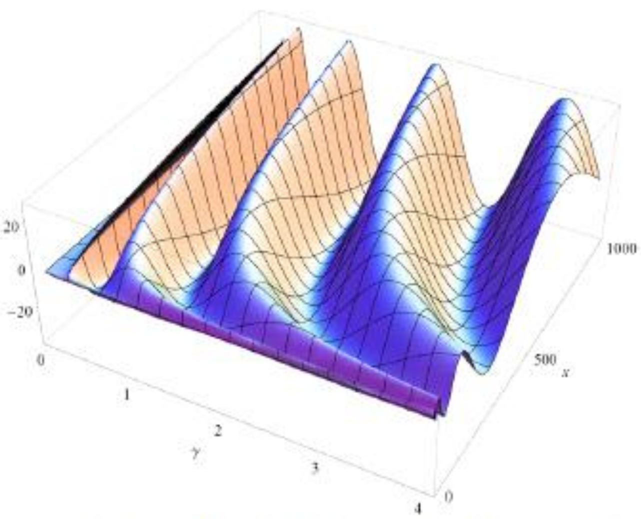
$$\text{Out[4]= } \frac{1 + 4\gamma}{4 z^2}$$

Example 5.3. Solutions of Euler's equation $x^2 y'' + \gamma y = 0$ oscillate for $\gamma > \frac{1}{4}$ but do not oscillate for $\gamma \le \frac{1}{4}$:

```
Rasterize[DensityPlot[
  If[γ < 1/4, x^((1-√(1-4γ))/2),
   √x Cos[Log[x] √(4γ-1)]],
  {γ, 0, 4}, {x, 0, 1000},
  PlotPoints → 100, LabelStyle → Medium
 ], ImageSize → 360]
```




```
Rasterize[Plot3D[
  If[γ < 1/4, x^((1-√(1-4γ))/2), √x Cos[Log[x] √(4γ-1)]],
  {γ, 0, 4}, {x, 0, 1000},
  PlotPoints → 100, AxesLabel → Automatic,
  LabelStyle → Medium
 ], ImageSize → 500]
```



Corollary 5.4. Suppose there exist numbers $\gamma > \frac{1}{4}$ and A such that for all $x \ge A$ we have $q(x) > \frac{\gamma}{x^2}$. Then every solution of $y'' + qy = 0$ oscillates infinitely often for $x > A$. However ~~if there is $\gamma < \frac{1}{4}$ such that~~ for all $x \ge A$ we have $q(x) \le \frac{\gamma}{4x^2}$, then solutions of $y'' + qy = 0$ have at most one zero for $x \ge A$.

4.2. **Changing the Independent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $z = \nu(x)$, where ν satisfies $\nu'' + p\nu' = 0$, then the equation becomes

$$\frac{d^2y}{dz^2} + Q(z)y = 0, \qquad \text{where} \qquad Q(z) = \frac{q(x(z))}{[\nu'(x(z))]^2}.$$

The zeros of y get moved by this transformation, so studying the oscillatory behaviour of $y(x)$ as $x \to \infty$ corresponds to studying the oscillatory behaviour of $y(z)$ as $z \to \lim_{x\to\infty} \nu(x)$, and the latter point may or may not be ∞. Note though, that the amplitudes of oscillations (if they occur), are unchanged.

done line

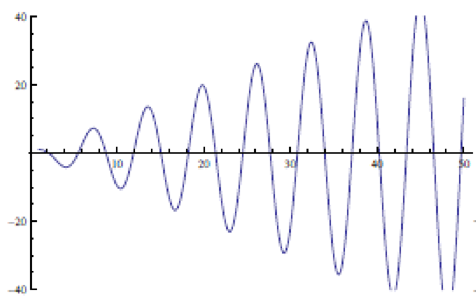
Example 4.2. Under the change of independent variable $z(x) = x^3/3$, the equation $y'' - \frac{2}{x}y' + y = 0$ becomes the equation $\frac{d^2y}{dz^2} + \frac{1}{(3z)^{4/3}}y = 0$:

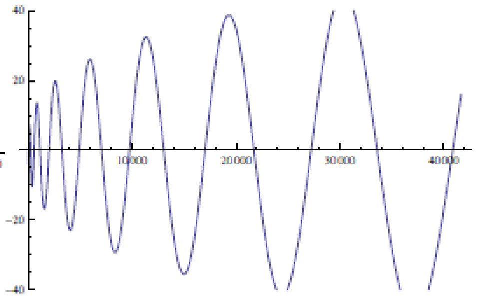
Indeed, $\nu'' - \frac{2}{x}\nu' = 0 \Rightarrow \nu' = x^2 \Rightarrow z = \nu = x^3/3$,

So $x = \sqrt[3]{3z}$ so $Q = \frac{1}{((\sqrt[3]{3z})^2)^2} = \frac{1}{(3z)^{4/3}}$

```
a = 50; b = 40;
ψ = NDSolve[
    y''[x] - 2/x y'[x] + y[x] == 0
     && y[1] == 1 && y'[1] == 0,
    y[x], {x, 1, a}
  ];
Plot[Evaluate[y[x] /. ψ],
  {x, 1, a}, PlotRange → {-b, b}]
```

```
ψ = NDSolve[
    y''[z] + 1/(3 z)^(4/3) y[z] == 0
     && y[1/3] == 1 && y'[1/3] == 0,
    y[z], {z, 1, a^3/3}
  ];
Plot[Evaluate[y[z] /. ψ],
  {z, 1, a^3/3}, PlotRange → {-b, b}]
```





Theorem 6.1. *Consider a solution y of the equation $y'' + py' + qy = 0$. If $q > 0$ and $q' + 2pq > 0$ on some interval $[a, b]$ and $y'(a) = 0 = y'(b)$, then $|y(a)| > |y(b)|$. If instead $q' + 2pq < 0$ and $y'(a) = 0 = y'(b)$, then $|y(a)| < |y(b)|$. Similarly for non-strict inequalities.*

Proof. Consider $F = y^2 + \frac{(y')^2}{q}$ and note that $F' = -(q' + 2pq)\frac{(y')^2}{q^2}$. □

Example 6.1. For Bessel's equation $y'' + \frac{1}{x}y' + (1 - \alpha^2/x^2)y = 0$ we have $q' + 2pq = 2/x > 0$, and hence the amplitudes of its oscillations decreases on $x > 0$. Yet for $y'' + y/x^2 = 0$ we have $q' + 2pq = \frac{-2}{x^3} < 0$, and hence the amplitudes of its oscillations increases on $x > 0$.