267ODEs-121126, Hour 31: Regular singular points, examples and the harder cases

November-25-12
3:40 PM

$Ly = x^2y'' + xpy' + qy = 0, \quad p(x) = \sum p_n x^n, \quad q(x) = \sum q_n x^n$

Try $x^\alpha \sum a_n x^n = \sum a_n x^{n+\alpha}$ $(a_0 = 1)$,

get $F(\alpha+n)\, a_n = -\sum_{k=0}^{n-1} a_k \left[(\alpha+k)p_{n-k} + q_{n-k}\right]$

where $F(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0$, the "indicial poly".

"The Fundamental Series": Ignore the coeff. of x^α, get $\phi_\alpha = \sum a_n(\alpha)\, x^{n+\alpha}$

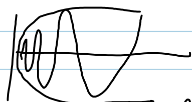
1. $L\phi_\alpha = F(\alpha)\, x^\alpha$
2. The coeff $a_n(\alpha)$ has at most simple poles at each $\alpha_1 + k, \alpha_2 + k, \; 0 < k \leq n$

The easy case: F has two roots $\alpha_1 > \alpha_2$, s.t. $\alpha_1 - \alpha_2 \notin \mathbb{N}$.

Sol'ns: $y_1 = \phi_{\alpha_1} \quad y_2 = \phi_{\alpha_2}$

<u>Example</u>. $y'' + \left(\frac{1}{2x^2} + \frac{1}{2(1-x^2)}\right)y = 0$

$F(\alpha) = \alpha(\alpha-1) + \frac{1}{2} = 0 \quad \alpha = \frac{1}{2} \pm \frac{1}{2}i$,

Sol'ns look like 

$\sqrt{x}\left(A\sin\left(\tfrac{1}{2}\log x\right) + B\cos\left(\tfrac{1}{2}\log x\right)\right)$

Now follow the FrobeniusSeries mathematica notebook alternating theory & practice.

<u>Exercise</u> In the $\alpha_1 = \alpha_2$ case,

$$y_2 = y_1 \log x + x^\alpha \sum_{n=1}^{\infty} b_n x^n$$

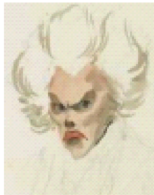
In the $\alpha_1 - \alpha_2 \in \mathbb{N}_{>0}$ case,

$$y_2 = c\, y_1 \log x + x^{\alpha_2}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)$$

Exercise 2.2. Using the change of variable $t = 1/x$, study the behaviour of Legendre's equation of order α,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

for large x and for *all* real α.



Adrien-Marie Legendre

Soln: $t = 1/x$, $1/x = t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} = -t^2\dot{y}$$

$$y'' = -t^2(-t^2\dot{y})^{\cdot} = t^4\ddot{y} + 2t^3\dot{y}$$

Legendre: $\left(1-\frac{1}{t^2}\right)(t^4\ddot{y} + 2t^3\dot{y}) - \frac{2}{t}(-t^2)\dot{y} + \alpha(\alpha+1)y = 0$

$$t^2(t^2-1)\ddot{y} + 2t(t^2-1+1)\dot{y} + \alpha(\alpha+1)y = 0$$

$$t^2\ddot{y} + t\frac{2t^2}{t^2-1} + \frac{\alpha(\alpha+1)}{t^2-1}y = 0$$

$F(r) = r(r-1) + \alpha(\alpha+1)$; $r_{1,2} = -\alpha, \alpha+1$

$y_1 = t^{\alpha+1}(1 + \sum a_n t^n)$ by symmetry, may as well assume $\alpha \geq -\frac{1}{2}$

$y_2 \sim t^{-\alpha}$

$\Rightarrow$ if $-1 < \alpha < 1$ bndd near ∞

if $\alpha > 0$ there are unbndd soln's.