Topic: Regular Singular Points.

This is a summary of http://drorbn.net/dbnvp/12-267-121120-2.php

Comments. 1. Why care about RSP?
2. The word "analytic" was not defined.

$\sum_{k=0}^{n} P_k y^{(k)} = 0$ If all P_k are analytic, "ordinary pt".

If $x^{n-k} P_k$ is analytic — "regular singular pt."

E.g. $x^2 y'' + x y' = y$ RSP

$x^3 y' = (x+1) y$ ISP (irreg. sing. pt.)

"Frobenius Series": Sol'n of the form

$x^\alpha A(x)$ where A is "analytic"

Example: $y'' + \frac{y}{4x^2} = 0$ $r(r-1) = -\frac{1}{4}$

$r = \frac{1}{2}$

No solutions w/ honest P.S.,

try $x^\alpha \sum a_n x^n$. Get → better assume $a_0 \neq 0$.

$\left((n+\alpha)(n+\alpha-1) + \frac{1}{4}\right) a_n = 0$ ...

$\Rightarrow y(x) = x^\alpha a_0 = a_0 \sqrt{x}$ 2^{nd} Sol?

Study $y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$ has RSP @ 0.

$p(x) = \sum p_n x^n$ $q(x) = \sum q_n x^n$

Try $y(x) = \sum a_n x^{n+\alpha}$

$y' = \dots$ $y'' = \dots$

$\therefore$ The coeff. of x^α is $\underbrace{\left(\alpha(\alpha-1) + p_0 \alpha + q_0\right)}\, a_0$

indicial polynomial.

.... get a "convolution" equation for the coeffs:

$$x^{n+\alpha}: \; a_n(\alpha+n)(\alpha+n-1) + \sum_{k=0}^{n} p_{n-k} a_k (\alpha+k) + \sum_{k=0}^{n} q_{n-k} a_k = 0$$

$$P(\alpha+n)\, a_n = -\sum_{k=0}^{n-1} a_k \left[(\alpha+k) p_{n-k} + q_{n-k} \right]$$

$\Rightarrow$ Problems if $\alpha+n$ is a root of the indicial polynomial!

$\alpha_1 > \alpha_2$

$y_1(x) = x^{\alpha_1}(- \ldots)$ works, always.

"you should fail the course if you can't find one solution; if you don't know the rest, you don't deserve to fail the course."

Case 1: $\alpha_1 \neq \alpha_2 + n$ for any n, $\alpha_1 \neq \alpha_2$

$y_2(x) = x^{\alpha_2}(- - -)$

Case 2: $\alpha_1 = \alpha_2 + N$, yet $0 = \sum_{k=0}^{N-1} a_k \left[(\alpha+k) p_{n-k} + q_{n-k} \right]$,

Then a_n is free

.... again a Frobenius-series soln.

Otherwise the 2nd sol'n is not a Frob series!

Case 3: $\alpha_1 = \alpha_2$: $y(x) = y_1(x) = x^{\alpha_1}(\ldots)$

Set $y(x,\alpha) = x^{\alpha}\left(\sum_{n=0}^{\infty} a_n(\alpha) x^n \right)$ ...

Then with $L = \frac{d^2}{dx^2} + \frac{p}{x}\frac{d}{dx} + \frac{q}{x^2}$, better mult. by x^2!

$$(Ly)(x,\alpha) = a_0 x^{\alpha-2} p(\alpha)$$

Take $\frac{\partial}{\partial \alpha}$ at $\alpha = \alpha_1$; RHS is 0; LHS

gives $$y_2(x) = (\log x) y_1(x) + \sum_{n=0}^{\infty} b_n x^{n+\alpha_1}$$

Case 4: $\alpha_1 = \alpha_2 + N$, $\quad 0 \neq \sum_{k=0}^{N-1} a_k \left[(\alpha+k) p_{n-k} + q_{n-k}\right]$

Not done!