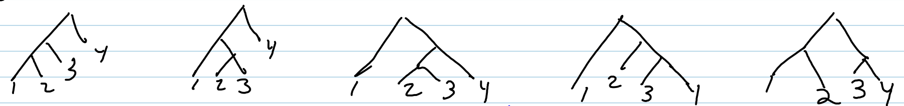
Riddle Along. How many binary trees w/ n leafs are there?

$n=4$:



<u>Claim</u> If $\Psi'(t)=A(t)\Psi(t)$, then either Ψ is regular for all t or singular for all t.

<u>pf 1</u>: Use existence & uniqueness

Debts: 1. Make ex/un interval length explicit.
2. pf2 using the "Wronskian" & det'.

Postponed.

<u>claim</u> Solutions to $V'(t)=A(t)V(t)$ exist and are unique wherever $A(t)$ is continuous.

Proof Lipshitz is non-issue. Usual existence/uniq. limited to $\frac{b}{M}$; if $|A_{ij}|<\mu$ on $I=[-1000,1000]$, then $M<\mu n b$ on I so $\frac{b}{M}>\frac{1}{\mu n}=\epsilon$ so wherever a sol'n exists, it exists an ϵ further.

<u>pf 2</u> Use the Wronskian $W=\det\Psi(t)$:

$$W(t+\epsilon)=\det(\Psi(t+\epsilon))=\det(\Psi(t)+\epsilon\Psi')=\det(\Psi+\epsilon A\Psi)$$

$$=\det(I+\epsilon A)\det\Psi=(1+\epsilon(\operatorname{tr}A))W$$

So $W'=(\operatorname{tr}A)\cdot W$ So $W=\exp\int \operatorname{tr}(A)\,dt\cdot W(0)$

Power Series, an unusual motivation.

1. Power series are keepers of combinatorial information.
2. Recursion relations $\Longleftrightarrow$ differential eqn's.

$A_n=\frac{1}{n+1}\binom{2n}{n} \longrightarrow F(x)=\sum A_n x^n$

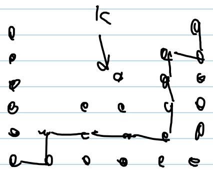
$C_n=\left(\begin{array}{c}\text{\# of below}\\ \text{-diagonal paths}\\ \text{from }(0,0)\text{ to }(n,n)\end{array}\right) \longrightarrow G(x)=\sum C_n x^n$

"Catalan numbers"

Goal: compute $G(x)$ ...

(from $(0,0)$ to (n,n))

"Catalan numbers"



$C_0 = 1$

Goal: compute $G(x)$ using algebra & $F(x)$ using ODEs, find $G=F$ & deduce $A_n = C_n$

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}$$

multiply by x^{n+1}, sum from $n=0$, get

$$G - 1 = x G^2$$

$$x G^2 - G + 1 = 0 \qquad G = \frac{1-\sqrt{1-4x}}{2x}$$

Now analyze F: $A_n = \frac{(2n)!}{n!(n+1)!}$

$$A_{n+1} = \frac{(2n+2)(2n+1)}{(n+1)(n+2)} A_n = 2\frac{2n+1}{n+2} A_n$$

$$(n+2) A_{n+1} = (4n+2) A_n$$

multiply by x^{n+1}, sum from $n=0$

$$\sum_{n=0}^{\infty} (n+1) A_n x^n - 1 = \sum_{n=0}^{\infty} (n+2) A_{n+1} x^{n+1} = x \sum_{n=0}^{\infty} (4n+2) A_n x^n$$

$$xF' + F - 1 = x(4xF' + 2F)$$

$$F(0) = 1 \qquad 0 = (4x^2 - x)F' + (2x-1)F + 1$$

Thus $F = G$ &

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad !$$

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f = (1 - Sqrt[1 - 4 x])/(2 x)
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$$\frac{1-\sqrt{1-4x}}{2x}$$

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Series[f, {x, 0, 10}]
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$1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + 4862x^9 + 16796x^{10} + O[x]^{11}$

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x (4 x - 1) D[f, x] + 2 x f - f + 1 // Simplify
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