Remember, remember!
The fifth of November,
The Gunpowder treason and plot;
I know of no reason
Why the Gunpowder treason
Should ever be forgot!

Read Along:
BDP chapter 5

Pasted from <http://www.potw.org/archive/potw405.html>

The non-homogeneous case, using diagonalization.

$V' = AV + g(t)$ set $V = C\cdot u$ $u = C^{-1}V$

$Cu' = ACu + g(t)$

$u' = C^{-1}ACu + C^{-1}g(t) = Du + C^{-1}g$

if $C^{-1}AC \overset{=D}{}$ is diagonal, this is a decoupled system.

Example $V' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} V + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$

$D = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}$ $C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ $C^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$u' = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix} u + \frac{1}{5}\begin{pmatrix} 5/t + 8 \\ 4 \end{pmatrix}$

$u_1' = 1/t + \frac{8}{5}$ $u_1 = \log t + \frac{8}{5}t + c_1$

$u_2' = -5u + 4/5$ $u_2 = \frac{4}{25} + c_2 e^{-5t}$

$V = C\cdot u = \begin{pmatrix} -8/25 + \frac{8}{5}t + \log t \\ 4/25 + \frac{16t}{5} + 2\log t \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

The non-homogeneous case, using a "fundamental matrix".

"Fundamental matrix for $V' = AV + g$" (A, g time dependent)

– a matrix whose columns are lin. indep. sol'ns of $V' = AV$

$\Longleftrightarrow$ $\Psi(t)$ invertible, w/ $\Psi'(t) = A(t)\Psi(t)$

Claim If $\Psi'(t) = A(t)\Psi(t)$, then either Ψ is regular for all t or singular for all t.

Pf 1: Use existence & uniqueness

Pf 2 Use the Wronskian $W = \det \Psi(t)$: not done

$$W(t+\epsilon) = \det(\Psi(t+\epsilon)) = \det(\Psi(t) + \epsilon\Psi') = \det(\Psi + \epsilon A\Psi)$$

$$= \det(1+\epsilon A)\det\Psi = (1+\epsilon(\mathrm{tr} A))W$$

So $W' = (\mathrm{tr} A)\cdot W$ So $W = \exp\int \mathrm{tr}(A)\,dt \cdot W(0)$

Set $v = \Psi\cdot u$, get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$v' = Av + g \longrightarrow \cancel{\Psi' u} + \Psi u' = \cancel{A\Psi u} + g$$

$$\Psi u' = g \Rightarrow u' = \Psi^{-1} g \Rightarrow u = \int \Psi^{-1} g\,dt$$

$$\Rightarrow v = \Psi\int(\Psi^{-1}g)\,dt$$

Example. $v' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} v + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$ $\Psi = \begin{pmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{pmatrix}$

$$\Psi^{-1} = e^{5t}\cdot\frac{1}{5}\cdot\begin{pmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2e^{5t} & e^{5t} \end{pmatrix}$$

In[7]:= $\Psi = \begin{pmatrix} 1 & -2\,E^{-5t} \\ 2 & E^{-5t} \end{pmatrix}$; Inverse[Ψ] // MatrixForm

Out[7]//MatrixForm=

$$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2\,e^{5t}}{5} & \frac{e^{5t}}{5} \end{pmatrix}$$

In[8]:= Inverse[Ψ].{t^{-1}, 2 t^{-1} + 4} // Simplify

Out[8]= $\left\{\frac{8}{5} + \frac{1}{t}, \frac{4\,e^{5t}}{5}\right\}$

In[9]:= Integrate[Inverse[Ψ].{t^{-1}, 2 t^{-1} + 4}, t]

Out[9]= $\left\{\frac{8\,t}{5} + \text{Log}[t], \frac{4\,e^{5t}}{25}\right\}$

In[10]:= Ψ.Integrate[Inverse[Ψ].{t^{-1}, 2 t^{-1} + 4}, t] // Expand

Out[10]= $\left\{-\frac{8}{25} + \frac{8\,t}{5} + \text{Log}[t], \frac{4}{25} + \frac{16\,t}{5} + 2\,\text{Log}[t]\right\}$