267ODEs-121030, Hour 22: Computing Matrix Exponentials

October-29-12
11:09 AM

HW5 is on web!
TT results: Discussion at end.

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

1. Converges
2. $e^0 = I$
3. $\exp(\mathrm{diag}(\lambda_i)) = \mathrm{diag}(e^{\lambda_i})$
4. $AB = BA \Rightarrow e^{A+B} = e^A e^B$
5. $e^{(t+s)A} = e^{tA} e^{sA}$
6. $\frac{d}{dt} e^{tA} = A e^{tA}$
7. $e^{C^{-1}DC} = C^{-1} e^D C$

on board.

Example Solve

$$\dot{x} = 4x - 6y \qquad x(0) = 2$$
$$\dot{y} = 3x - 5y \qquad y(0) = -1$$

```
In[1]:= D1 = (1 0; 0 -2); D2 = (2 1; 0 2); CC = (1 -1; -1 2);
        Inverse[CC] // MatrixForm
Out[2]//MatrixForm=
(2 1
 1 1)

In[3]:= MatrixForm /@ {D1, CC.D1.Inverse[CC]}
Out[3]= {(1 0; 0 -2), (4 3; -6 -5)}

In[4]:= MatrixForm /@ {D2, CC.D2.Inverse[CC]}
Out[4]= {(2 1; 0 2), (3 1; -1 1)}
```

In general, diagonalization works at least when the characteristic poly. has n distinct roots

Thm (Jordan canonical form) If $T: V \to V$ is a linear transformation [over $\mathbb{C}$], then there is a basis $\beta = (v_1, \dots, v_n)$ of V s.t.

$$D = [T]_\beta = \begin{pmatrix} \boxed{B_1} & 0 & 0 \\ 0 & \boxed{B_2} & 0 \\ 0 & 0 & \boxed{B_3} \end{pmatrix} \qquad B_i = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Exponentiate $(\lambda I + J)$

Example solve

$$\dot{x} = 2x - y \qquad x(0) = 2$$

$$\dot{x} = 3x - y \qquad x(0) = 2$$
$$\dot{y} = x + y \qquad y(0) = -1$$

Complex eigenvalues in tutorial. done line

Phase portraits: First philosophy, then follow handout.

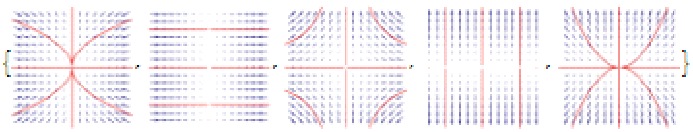
Distribute TT at end!

Pensieve header: Plotting Phase Profiles.

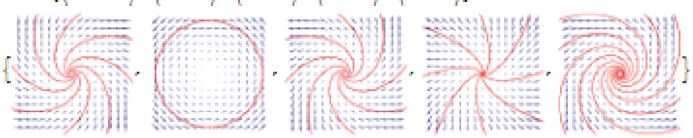
```
PP[A_] := Show[VectorPlot[A.(x
                              y), {x, -1, 1}, {y, -1, 1}, Frame → None],
  ParametricPlot[Table[MatrixExp[t A].(Cos[θ]
                                       Sin[θ]), {θ, π/4, 2π, π/4}],
   {t, -π, π}, ColorFunction → (Red &)],
  ImageSize → 150]
```

PP /@ $\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$

PP /@ $\left\{\begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\right\}$



PP /@ $\left\{\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}3 & -1\\ 1 & 3\end{pmatrix}, \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\right\}$



PP /@ $\left\{\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\}$