TT. Friday Oct. 26 9-10 @ GB404

.... I'll try to have a sample test for Friday.

Meanwhile, concentrate on <u>understanding everything</u>

Riddle Along. Is there a cont. $f: [0,1] \to \mathbb{R}$,

$f(0)=0$, $f(1)=1$, f is constant on intervals $[a_i, b_i]$

w/ $\sum (b_i - a_i) = 1$?

שיטות נומריות:

1. שימוש במשפט פיקרד: אין מעשי! דוגמה $y'=-y$, $y(0)=1$

2. שיטת אוילר

x_0 $\quad x_{n+1}=x_n+h$

y_0 $\quad y_{n+1}=y_n+hy'_n \;(=y_n+hf(x_n,y_n))$

דוגמה - תרגיל $y(1)$ ל- $h=1,\frac{1}{2},\frac{1}{3}$

$y(1)=0.3679$

$h=1\to 0$

$h=\frac{1}{2}\to\frac{1}{4}=0.25$

$h=\frac{1}{3}\to 8/27\approx 0.2963$

כמה מהר מגיעים לתוצאה?

שגיאה מקומית: (מניחים y_n מדויק)

$\phi(x_n+h)=\phi(x_n)+h\phi'(x_n)+O(h^2)$

$y_{n+1}=y_n+hy'_n$ $\quad O(h^2)$

שגיאה מצטברת:

תשובה [illegible]: $E\sim\frac{x-x_0}{h}\cdot h^2\propto h$

שיקולי "דיוק דריכה" [illegible] התוצאה.

מסקנה - כדי להגיע לדיוק 10^{-6} צריך 10^{9} חישובים - וזה כל המספר. $x\sim 1{,}000$ עד

3. שיטת אוילר משופרת: $x_{n+1}=x_n+h$,

$\phi(x_{n+1})=\phi(x_n)+\int_{x_n}^{x_{n+1}}f(x,\phi(x))dx$

$y_{n+1}=y_n+\frac{f(x_n,y_n)+f(x_{n+1},y_{n+1})}{2}h=y_n+\frac{y'_n+f(x_n+h,y_n+hy'_n)}{2}h$

בטכניקה אחרת: נתונים (x_n,y_n)

$k_1=f(x_n,y_n)$

$k_2=f(x_n+h,y_n+k_1h)$

$y_{n+1}=y_n+\frac{k_1+k_2}{2}h$

start line

הערכת שגיאה: מקומית $\sim h^3$

מצטברת $\sim h^2$ לדיוק 10^{-6} צריך $2\cdot10^3$,

4. שיטה (כללית) לשיטות נוספות:

$k_1=f(x_n,y_n)$

$k_2=f(x_n+\alpha_2h,y_n+\alpha_2k_1h)$

$k_3=f(x_n+\alpha_3h,y_n+\alpha_3k_2h)$

$k_7=\dots$

$y_{n+1}=y_n+(\beta_1k_1+\dots+\beta_7k_7)h$

[illegible] ובוחרים α_i,β_i כך [illegible]

5. שיטת Runge-Kutta

אז
$$\begin{cases}k_1=f(x_n,y_n)\\ k_2=f(x_n+\frac{1}{2}h,y_n+\frac{1}{2}hk_1)\\ k_3=f(x_n+\frac{1}{2}h,y_n+\frac{1}{2}hk_2)\\ k_4=f(x_n+h,y_n+hk_3)\end{cases}$$

$y_{n+1}=y_n+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$

מקומית $\sim h^5$

מצטברת $\sim h^4$

Now play with LocalRungeKutta-2.nb.

& with NumericalMethodsComparison.nb

Penseive header: Checking the local error in Runge-Kutta, more efficiently.

In[1]:= $\phi[x_0] = y_0;$

$\phi_0[x\_] := \phi[x];$

$\phi_{k\_}[x\_] \ /; \ k \geq 1 := \phi_k[x] = \text{Expand}[$
$\quad \partial_x(\phi_{k-1}[x]) \ /. \ \phi_0'[x] \mapsto f[x, \phi_0[x]]$
$];$

$$\text{ser1} = \sum_{k=0}^{4} \frac{1}{k!} \phi_k[x]\, h^k \ /. \ x \to x_0$$

Out[4]= $$h f[x_0, y_0] + y_0 + \frac{1}{2} h^2 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0] + f^{(1,0)}[x_0, y_0]\right) + \frac{1}{6} h^3 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^2 + f[x_0, y_0]^2 f^{(0,2)}[x_0, y_0] + f^{(0,1)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 2 f[x_0, y_0] f^{(1,1)}[x_0, y_0] + f^{(2,0)}[x_0, y_0]\right) + \frac{1}{24} h^4 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^3 + 4 f[x_0, y_0]^2 f^{(0,1)}[x_0, y_0] f^{(0,2)}[x_0, y_0] + f[x_0, y_0]^3 f^{(0,3)}[x_0, y_0] + f^{(0,1)}[x_0, y_0]^2 f^{(1,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(0,2)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 5 f[x_0, y_0] f^{(0,1)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f^{(1,0)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f[x_0, y_0]^2 f^{(1,2)}[x_0, y_0] + f^{(0,1)}[x_0, y_0] f^{(2,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(2,1)}[x_0, y_0] + f^{(3,0)}[x_0, y_0]\right)$$

In[5]:= $k_1 = h f[x_0, y_0];$

$k_2 = h f\left[x_0 + \frac{h}{2}, \ y_0 + \frac{1}{2} k_1\right];$

$k_3 = h f\left[x_0 + \frac{h}{2}, \ y_0 + \frac{1}{2} k_2\right];$

$k_4 = h f[x_0 + h, \ y_0 + k_3];$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2 k_2 + 2 k_3 + k_4)$

Out[9]= $$\frac{1}{6}\left(h f[x_0, y_0] + 2 h f\left[\frac{h}{2} + x_0, \frac{1}{2} h f[x_0, y_0] + y_0\right] + 2 h f\left[\frac{h}{2} + x_0, \frac{1}{2} h f\left[\frac{h}{2} + x_0, \frac{1}{2} h f[x_0, y_0] + y_0\right] + y_0\right] + h f\left[h + x_0, h f\left[\frac{h}{2} + x_0, \frac{1}{2} h f\left[\frac{h}{2} + x_0, \frac{1}{2} h f[x_0, y_0] + y_0\right] + y_0\right] + y_0\right]\right) + y_0$$

In[10]:= ser2 = Series[y_1, {h, 0, 4}] // Normal

Out[10]= $$h f[x_0, y_0] + y_0 + \frac{1}{2} h^2 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0] + f^{(1,0)}[x_0, y_0]\right) + \frac{1}{6} h^3 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^2 + f[x_0, y_0]^2 f^{(0,2)}[x_0, y_0] + f^{(0,1)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 2 f[x_0, y_0] f^{(1,1)}[x_0, y_0] + f^{(2,0)}[x_0, y_0]\right) + \frac{1}{24} h^4 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0]^3 + 4 f[x_0, y_0]^2 f^{(0,1)}[x_0, y_0] f^{(0,2)}[x_0, y_0] + f[x_0, y_0]^3 f^{(0,3)}[x_0, y_0] + f^{(0,1)}[x_0, y_0]^2 f^{(1,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(0,2)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 5 f[x_0, y_0] f^{(0,1)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f^{(1,0)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 3 f[x_0, y_0]^2 f^{(1,2)}[x_0, y_0] + f^{(0,1)}[x_0, y_0] f^{(2,0)}[x_0, y_0] + 3 f[x_0, y_0] f^{(2,1)}[x_0, y_0] + f^{(3,0)}[x_0, y_0]\right)$$

In[11]:= ser1 == ser2

Out[11]= True

Pensieve header: A comparison of Euler, Improved Euler, and Runge-Kutta on $y'=-y$.

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In[12]:= Euler[f_, x0_, y0_, x_, n_, p_] := Module[
           {h = N[(x - x0) / n, p], xj = N[x0, p], yj = N[y0, p], k1},
           Do[
            k1 = f[xj, yj];
            xj = xj + h;
            yj = yj + h k1,
            {n}
           ];
           yj
          ]

In[13]:= f1[x_, y_] := -y;
         {Euler[f1, 0, 1, 1, #, 10] & /@ {1, 10, 10^2, 10^3, 10^4, 10^5}, N[1/E, 10]} //
          Timing

Out[14]= {0.406,
          {{0. × 10^-10, 0.348678440, 0.366032341, 0.367695425, 0.367861046, 0.367877602}, 0.3678794412}}

In[15]:= ImprovedEuler[f_, x0_, y0_, x_, n_, p_] := Module[
           {h = N[(x - x0) / n, p], xj = N[x0, p], yj = N[y0, p], k1, k2},
           Do[
            k1 = f[xj, yj];
            k2 = f[xj + h, yj + h k1];
            xj = xj + h;
            yj = yj + h (k1 + k2) / 2,
            {n}
           ];
           yj
          ]

In[16]:= ImprovedEuler[f1, 0, 1, 1, #, 10] & /@ {1, 10, 10^2, 10^3, 10^4, 10^5} // Timing

Out[16]= {0.92, {0.500000000, 0.368540985, 0.367885619, 0.367879503, 0.367879442, 0.367879441}}

In[17]:= RungeKutta[f_, x0_, y0_, x_, n_, p_] := Module[
           {h = N[(x - x0) / n, p], xj = N[x0, p], yj = N[y0, p], k1, k2, k3, k4},
           Do[
            k1 = f[xj, yj];
            k2 = f[xj + h / 2, yj + h k1 / 2];
            k3 = f[xj + h / 2, yj + h k2 / 2];
            k4 = f[xj + h, yj + h k3];
            xj = xj + h;
            yj = yj + h (k1 + 2 k2 + 2 k3 + k4) / 6,
            {n}
           ];
           yj
          ]

In[18]:= {RungeKutta[f1, 0, 1, 1, #, 20] & /@ {1, 10, 10^2, 10^3, 10^4, 10^5}, N[1/E, 20]} //
          Timing

Out[18]= {2.387, {{0.37500000000000000000,
            0.36787977441249843340, 0.36787944120235551160, 0.36787944117144533898,
            0.36787944117144232190, 0.36787944117144232160}, 0.36787944117144232160}}
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