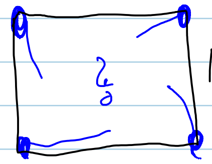
267ODEs-120914, Hour 3: Linear Equations

September-13-12
11:25 AM

Read Along. BDP chapter 1 & 2.1–2.2, 2.4–2.6

Riddle Along.



1m $v = 1\ m/min$
when will they meet?

Words: "Differential Equation"
"Ordinary" / "Partial"
"Order"

Type 0: $y' = f(x)$ Sol'n: $y = F(x) + C$
$= \int f(x)\,dx$

[often when we reduce an equation to this, we stop there]

Type 1: "1st order homogeneous linear":

$$a(x)\mu' + b(x)\mu = 0 \qquad (L\mu = 0)$$

or

$$\mu' = p(x)\mu$$

Sol'n: $\frac{\mu'}{\mu} = p(x) \Rightarrow (\log \mu)' = p(x) \Rightarrow \mu = e^{\int p\,dx}$

Example $t \cdot \dot{\mu} = 2\mu \Rightarrow \mu = e^{\int \frac{2}{t} dt} = e^{2\log t + c}$
$= c t^2$

Type 2: "1st order linear, not homogeneous"

$$a(x)y' + b(x)y = c(x) \qquad L\mu = c$$

or

$$y' + p \cdot y = g$$

trick: multiply by an "integrating factor" μ so that the LHS would be a derivative:

$$g\mu = \mu y' + p\mu y = (\nu y)' \qquad \nu = \mu \quad \nu' = p\mu$$

Example: Solve $ty' + 2y = 4t^2$ $\quad y(1) = 2$

Sol'n With the following (WTF) $\quad y' = -\frac{2}{t}y + 4t$

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In[1]:= VectorPlot[{1, -2/t y + 4 t}, {t, -2, 2}, {y, -2, 4},
  VectorPoints → 30,
  VectorScale → {Automatic, Automatic, None},
  VectorStyle → Arrowheads[0]]
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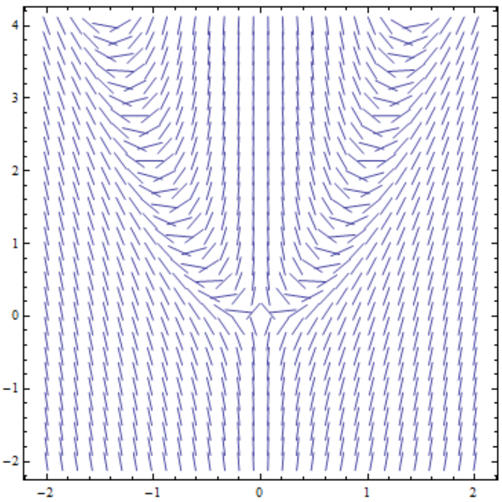
Out[1]=



$$(t\mu y)' = t\mu y' + 2y\mu \quad (= 4t^2\mu)$$

$$(t\mu)' = 2\mu$$

$$\mu + t\mu' = 2\mu \qquad t\mu' = \mu$$

$$t\frac{\mu'}{\mu} = 1 \qquad t(\log\mu)' = 1$$

$$(\log\mu)' = \frac{1}{t} \qquad \log\mu = \log t$$

$$\mu = t$$

$$(t^2 y)' = 4t^3$$

$$t^2 y = t^4 + C \qquad y = t^2 + \frac{C}{t^2} \qquad C = 1$$

Verify!

done line

Separable equations: $m(x) + n(y)\frac{dy}{dx} = 0$

4. הפרדת משתנים:

$m(x)+n(y)y'=0$

השיטה הקלה להצדקה:

שיש נגזרות $M(x)$ ו-$N(y)$ כך ש $M_x=m$ ו-$N_y=n$

$$M_x(x)+N_y(y)y'=0$$

$$\frac{d}{dx}M+\frac{d}{dx}N(y)=0$$

$$\frac{d}{dx}\left(M(x)+N(y)\right)=0$$

$$M(x)+N(y)=C$$

השיטה הנוחה לזכירה:

$$m(x)dx+n(y)dy=0$$

$$\int m(x)\,dx+\int n(y)\,dy=C$$

דוגמא: $y'=\frac{3x^2+4x+2}{2(y-1)}$, $y(0)=-1$

$$2(y-1)dy=(3x^2+4x+2)dx$$

$$y^2-2y=x^3+2x^2+2x+C \qquad C=3$$

$$y=1\pm\sqrt{x^3+2x^2+2x+4}$$

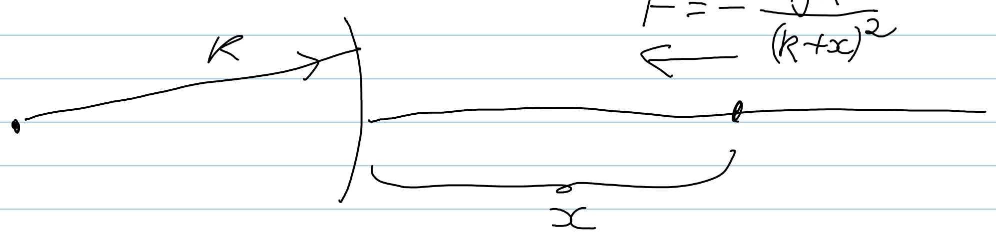
$(-)$

Example: The brachistochrone.

Example: Escape velocities

$$F=-\frac{mgR^2}{(R+x)^2}$$

scales right & correct at $x=0$.



So

$$\cancel{m}\frac{dv}{dt}=-\frac{\cancel{m}gR^2}{(R+x)^2}$$

$$\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\cdot\frac{dv}{dx}\implies v\frac{dv}{dx}=-\frac{gR^2}{(R+x)^2}$$

$$\frac{v^2}{2}=\frac{gR^2}{R+x}+C \qquad v(0)=v_0\implies C=\frac{v_0^2}{2}-gR$$

So $\frac{v^2}{2}=\frac{gR^2}{R+x}-gR+\frac{v_0^2}{2}$

For which v_0, $\lim_{x\to\infty}v(x)=0$?

$$gR=\frac{v_c^2}{2} \qquad v_c=\sqrt{2gR}$$