

Pensive header: Checking the local error in Runge-Kutta, more efficiently.

$$\phi[\mathbf{x}_0] = \mathbf{y}_0;$$

$$\phi_0[\mathbf{x}_-] := \phi[\mathbf{x}];$$

$$\phi_k[\mathbf{x}_-] \ /; \ k \geq 1 := \phi_k[\mathbf{x}] = \text{Expand}[\partial_x(\phi_{k-1}[\mathbf{x}]) \ /. \ \phi_0'[\mathbf{x}] \ \Rightarrow \ \mathbf{f}[\mathbf{x}, \phi_0[\mathbf{x}]]];$$

$$\text{ser1} = \sum_{k=0}^4 \frac{1}{k!} \phi_k[\mathbf{x}] h^k \ /. \ \mathbf{x} \rightarrow \mathbf{x}_0$$

$$h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{y}_0 + \frac{1}{2} h^2 \left(\mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{f}^{(1,0)}[\mathbf{x}_0, \mathbf{y}_0] \right) +$$

$$\frac{1}{6} h^3 \left(\mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0]^2 + \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0]^2 \mathbf{f}^{(0,2)}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$\mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(1,0)}[\mathbf{x}_0, \mathbf{y}_0] + 2 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(1,1)}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{f}^{(2,0)}[\mathbf{x}_0, \mathbf{y}_0] \right) +$$

$$\frac{1}{24} h^4 \left(\mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0]^3 + 4 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0]^2 \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,2)}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$\mathbf{f}[\mathbf{x}_0, \mathbf{y}_0]^3 \mathbf{f}^{(0,3)}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0]^2 \mathbf{f}^{(1,0)}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$3 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,2)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(1,0)}[\mathbf{x}_0, \mathbf{y}_0] + 5 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(1,1)}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$3 \mathbf{f}^{(1,0)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(1,1)}[\mathbf{x}_0, \mathbf{y}_0] + 3 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0]^2 \mathbf{f}^{(1,2)}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$\mathbf{f}^{(0,1)}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(2,0)}[\mathbf{x}_0, \mathbf{y}_0] + 3 \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] \mathbf{f}^{(2,1)}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{f}^{(3,0)}[\mathbf{x}_0, \mathbf{y}_0] \right)$$

$$\mathbf{k}_1 = h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0];$$

$$\mathbf{k}_2 = h \mathbf{f}\left[\mathbf{x}_0 + \frac{h}{2}, \mathbf{y}_0 + \frac{1}{2} \mathbf{k}_1\right];$$

$$\mathbf{k}_3 = h \mathbf{f}\left[\mathbf{x}_0 + \frac{h}{3}, \mathbf{y}_0 + \frac{1}{3} \mathbf{k}_2\right];$$

$$\mathbf{k}_4 = h \mathbf{f}[\mathbf{x}_0 + h, \mathbf{y}_0 + \mathbf{k}_3];$$

$$\mathbf{y}_1 = \mathbf{y}_0 + \frac{1}{6} (\mathbf{k}_1 + 2 \mathbf{k}_2 + 2 \mathbf{k}_3 + \mathbf{k}_4)$$

$$\frac{1}{6} \left(h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] +$$

$$2 h \mathbf{f}\left[\frac{h}{3} + \mathbf{x}_0, \frac{1}{3} h \mathbf{f}\left[\frac{h}{2} + \mathbf{x}_0, \frac{1}{2} h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{y}_0\right] + \mathbf{y}_0\right] + 2 h \mathbf{f}\left[\frac{h}{2} + \mathbf{x}_0, \frac{1}{2} h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{y}_0\right] +$$

$$h \mathbf{f}\left[h + \mathbf{x}_0, h \mathbf{f}\left[\frac{h}{3} + \mathbf{x}_0, \frac{1}{3} h \mathbf{f}\left[\frac{h}{2} + \mathbf{x}_0, \frac{1}{2} h \mathbf{f}[\mathbf{x}_0, \mathbf{y}_0] + \mathbf{y}_0\right] + \mathbf{y}_0\right] + \mathbf{y}_0\right] \right) + \mathbf{y}_0$$

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ser2 = Series[y1, {h, 0, 4}] // Normal
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$$\begin{aligned}
 & h f[x_0, y_0] + y_0 + \frac{4}{9} h^2 \left(f[x_0, y_0] f^{(0,1)}[x_0, y_0] + f^{(1,0)}[x_0, y_0] \right) + \\
 & \frac{1}{216} h^3 \left(24 f[x_0, y_0] f^{(0,1)}[x_0, y_0]^2 + 31 f[x_0, y_0]^2 f^{(0,2)}[x_0, y_0] + \right. \\
 & \quad \left. 24 f^{(0,1)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 62 f[x_0, y_0] f^{(1,1)}[x_0, y_0] + 31 f^{(2,0)}[x_0, y_0] \right) + \\
 & \frac{1}{3888} h^4 \left(108 f[x_0, y_0] f^{(0,1)}[x_0, y_0]^3 + 378 f[x_0, y_0]^2 f^{(0,1)}[x_0, y_0] f^{(0,2)}[x_0, y_0] + \right. \\
 & \quad 143 f[x_0, y_0]^3 f^{(0,3)}[x_0, y_0] + 108 f^{(0,1)}[x_0, y_0]^2 f^{(1,0)}[x_0, y_0] + \\
 & \quad 288 f[x_0, y_0] f^{(0,2)}[x_0, y_0] f^{(1,0)}[x_0, y_0] + 468 f[x_0, y_0] f^{(0,1)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + \\
 & \quad 288 f^{(1,0)}[x_0, y_0] f^{(1,1)}[x_0, y_0] + 429 f[x_0, y_0]^2 f^{(1,2)}[x_0, y_0] + \\
 & \quad \left. 90 f^{(0,1)}[x_0, y_0] f^{(2,0)}[x_0, y_0] + 429 f[x_0, y_0] f^{(2,1)}[x_0, y_0] + 143 f^{(3,0)}[x_0, y_0] \right)
 \end{aligned}$$

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TrueQ[ser1 == ser2]
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False
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