Penseive header: Finding the most general Runge-Kutta method.

```
n = 2; m = 2;

Clear[ϕ];
ϕ[x0] = y0;
ϕ0[x_] := ϕ[x];
ϕk_[x_] /; k ≥ 1 := ϕk[x] = Expand[
    ∂x (ϕk-1[x]) /. ϕ0'[x] :→ f[x, ϕ0[x]]
   ];
```

$$\text{ExactSeries} = \sum_{k=0}^{n} \frac{1}{k!} \phi_k[x]\, h^k \;/.\; x \to x_0 \;/.\; \{f[x_0, y_0] \to f_{0,0},\ f^{(i\_,j\_)}[x_0, y_0] :\to f_{i,j}\}$$

$$y_0 + h\, f_{0,0} + \frac{1}{2} h^2 (f_{0,0}\, f_{0,1} + f_{1,0})$$

$$k_1 = f[x_0, y_0];$$

$$\text{Table}\Big[k_j = f\Big[x_0 + h\,\gamma_j,\ y_0 + h \sum_{i=1}^{j-1} \alpha_{j,i}\, k_i\Big],\ \{j, 2, m\}\Big];$$

$$y_1 = y_0 + h \sum_{j=1}^{m} \beta_j\, k_j;$$

```
ApproximationSeries =
 (Series[y1, {h, 0, n}] // Normal // Collect[#, h, Simplify] &) /.
  {f[x0, y0] → f0,0, f^(i_,j_)[x0, y0] :→ fi,j}
```

$$y_0 + h\,(\beta_1 + \beta_2)\, f_{0,0} + h^2\, \beta_2\, (\gamma_2\, f_{1,0} + f_{0,0}\, f_{0,1}\, \alpha_{2,1})$$

```
Union[Cases[ExactSeries, f_,_, Infinity]]
```

$$\{f_{0,0},\ f_{0,1},\ f_{1,0}\}$$

```
sol = SolveAlways[
  ExactSeries == ApproximationSeries,
  Union[{h}, Cases[ExactSeries, f_,_, Infinity]]
 ]
```

$$\Big\{\Big\{\beta_1 \to 1 - \beta_2,\ \gamma_2 \to \frac{1}{2\beta_2},\ \alpha_{2,1} \to \frac{1}{2\beta_2}\Big\}\Big\}$$

```
sol = SolveAlways[
  ExactSeries == ApproximationSeries &&
```

$$\beta_1 == \frac{1}{6}\ \&\&\ \beta_2 == \frac{1}{3}\ \&\&\ \beta_3 == \frac{1}{3}\ \&\&\ \alpha_{3,1} == 0\ \&\&\ \alpha_{4,1} == 0\ \&\&\ \alpha_{4,2} == 0,$$

```
  Union[{h}, Cases[ExactSeries, f_,_, Infinity]]
 ]
```

{}