Qualitative Analysis

Based on a 1989 Princeton University handout by George Em Karniadakis.

This is a "first printing" and it is likely to contain many typos and other mistakes.

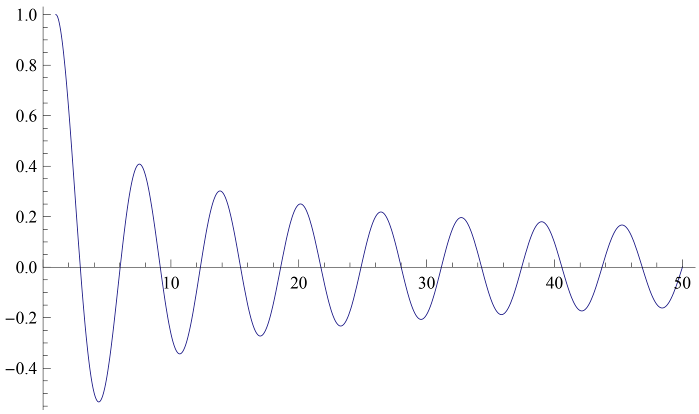
Contents

1. Introduction

Example 1.1. Bessel's equation of order 0:

```
{J₀} = NDSolve[
    x² y''[x] + x y'[x] + x² y[x] == 0
     && y[1] == 1 && y'[1] == 0,
    y[x], {x, 1, 50}
  ];
Plot[Evaluate[y[x] /. J₀], {x, 1, 50}]
```

$$x^2y'' + xy' + x^2y = 0$$

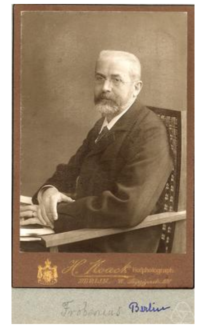


- Why does it oscillate?
- What does the "period" approach?
- What does the "amplitude" approach?

A power series, or a numerical approximation, won't help!

2. Regular Singular Points

Suppose 0 is a regular singular point of the equation



Ferdinand Georg Frobenius, 1849–1917, Oberwolfach image

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{1}$$

(Meaning simply that p and q above have a power series expansion around 0). Let $p_0 = p(0)$ and $q_0 = q(0)$, and let r_1 and r_2 be the roots of the indicial equation $r(r-1) + p_0 r + q_0 = 0$ (if they are real and distinct, assume also that $r_1 > r_2$). Then for $x > 0$ Equation (1) has two linearly independent solutions y_1 and y_2, such that

$$y_1 = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

and

$$y_2 = \begin{cases} y_1 \log x + x^r \sum_{n=1}^{\infty} b_n x^n & r_1 = r_2 = r \\ c y_1 \log x + x^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right) & r_1 - r_2 = N \in \mathbb{N}_{>0} \\ x^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right) & \text{otherwise.} \end{cases}$$

This can be used to deduce qualitative information! The behaviour near 0 of a power series is dominated by its 0th term. The cases are:

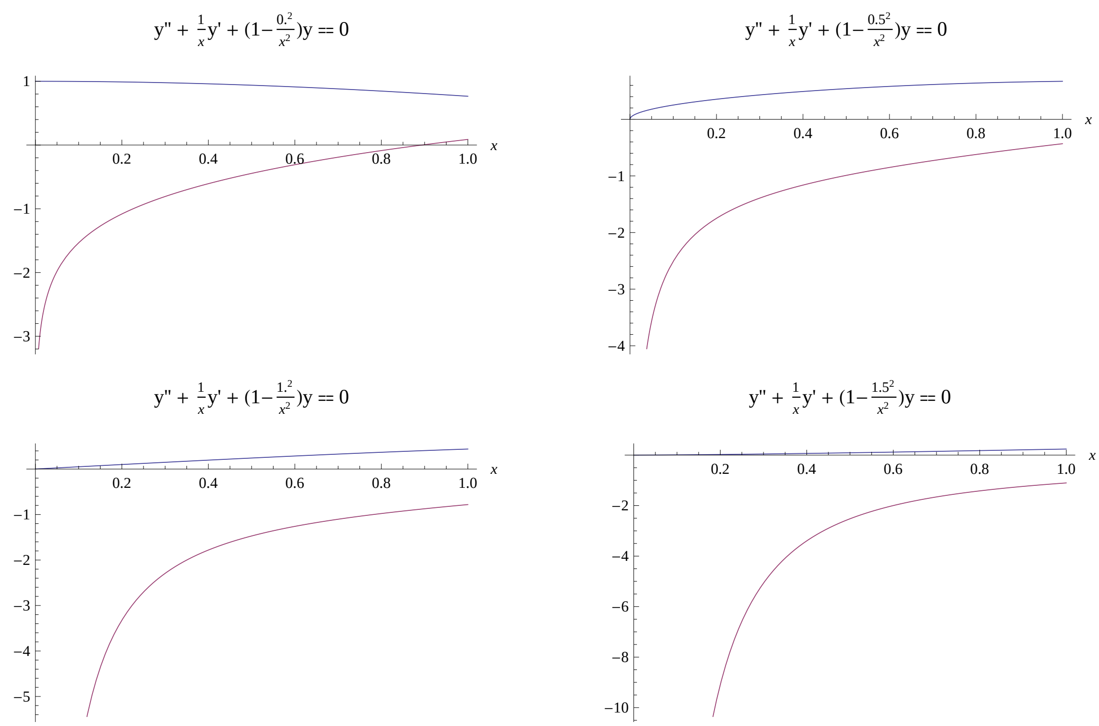
$$y \sim \begin{cases} a x^{r_1} + b x^{r_2} & r_1 - r_2 \in \mathbb{R} \setminus \mathbb{Z} \\ x^{\alpha}(a\cos(\beta \log x) + b \sin(\beta \log x)) & r_{1,2} = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R} \\ x^r (a + b\log x) & r_1 = r_2 = r \\ x^{r_1}(a + bc\log x) + b x^{r_2} & r_1 - r_2 \in \mathbb{N}_{>0} \end{cases}$$

Example 2.1. Bessel's equation of order α,

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\alpha^2}{x^2}\right) = 0,$$

has indicial equation $r(r-1) + r - \alpha^2 = 0$ whose solutions are $r_{1,2} = \pm\alpha$. Here are a few possibilities:

```
GraphicsGrid[Partition[Table[
   Plot[{BesselJ[α, x], BesselY[α, x]}, {x, 0, 1},
    AxesLabel → Automatic, PlotPoints → 100,
    PlotLabel → StringReplace["y'' + 1/x y' + (1-α^2/x^2) y == 0",
      "α" → ToString[α]]],
   {α, 0., 1.5, 0.5}
  ], 2]]
```

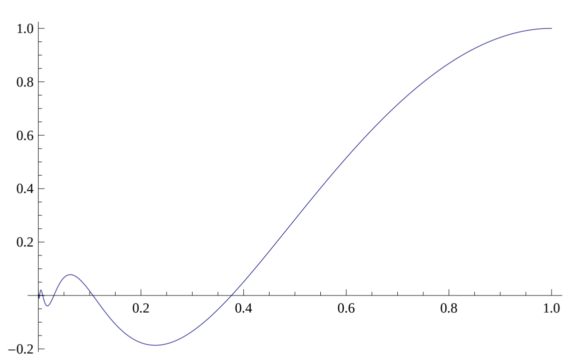


Example 2.2. The equation

$$y'' - 3y' + \left(\frac{13}{2x^2} + \cos x\right) y = 0$$

has $r_{1,2} = \frac{1}{2} \pm \frac{5}{2} i$.

```
Sol = NDSolve[
   y''[x] - 3 y'[x] + (13/(2 x^2) + Cos[x]) y[x] == 0 &&
    y[1] == 1 && y'[1] == 0,
   y[x], {x, ϵ = 10^-9, 1}
  ];
Plot[Evaluate[y[x] /. Sol], {x, ϵ, 1}, PlotPoints → 1000]
```



Exercise 2.1. Determine the behaviour near $x = 0$ of solutions of the equation

$$y'' + \left(\frac{1}{2x^2} + \frac{1}{2(1-x^2)}\right) y = 0.$$

Adrien-Marie Legendre

Exercise 2.2. Using the change of variable $t = 1/x$, study the behaviour of Legendre's equation of order α,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

for large x and for *all* real α.

Exercise 2.3. Find the general solution of Legendre's equation of order $\alpha = 0$,

(1) using power series, and,
(2) explicitly,

and determine the behaviour of these solutions as $x \to \infty$.

Exercise 2.4. Show that $x = 0$ is a regular singular point of the equation

$$x^3y'' + 2(1-\cos x)y' + (\sin x)y = 0$$

and study the qualitative behaviour of its solutions near that point.

Exercise 2.5. Show that for any non-zero value of the constant β, the point $x = \infty$ is a regular singular point of the equation

$$x^2y'' + 2xy' + \beta y = 0.$$

Study the behaviour of this equation near $x = \infty$ for $\beta = -\frac{3}{4}, \frac{3}{16}, \frac{1}{4}, \frac{5}{4}$. What if $\beta = 0$?

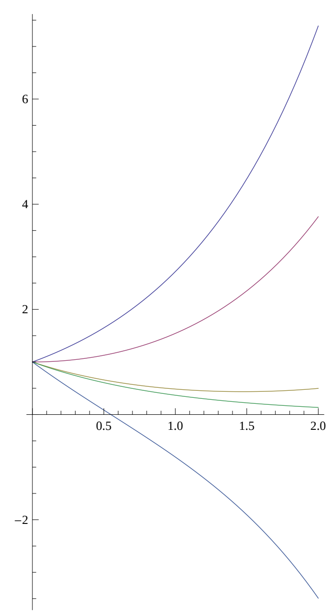
Exercise 2.6. Show that $x = \infty$ is *not* a regular singular point for the constant-coefficient equation $y'' + ay' + by = 0$ for any values of a and b (except $a = b = 0$).

3. The Basic Oscillation Theorems

Theorem 3.1. *If $q(x) < 0$ for every x in some connected subset I of $\mathbb{R}$, then any solution of $y'' + qy = 0$ may have at most one zero on I.*

Example 3.1. Consider the solutions of $y'' - y = 0$ with $y(0) = 1$ and $y'(0) = c$, for $c \in \{1, 0, -0.9, -1, -2\}$.

```
Plot[Evaluate[Table[
   y[x] /.
    DSolve[y''[x] - y[x] == 0
      && y[0] == 1 && y'[0] == c,
     y[x], x],
   {c, {1, 0, -0.9, -1, -2}}
  ]], {x, 0, 2}, AspectRatio → 2]
```



Exercise 3.1. Solve the equation $y'' + \frac{3}{16x^2}y = 0$, and decide if its solutions ever oscillate.

Theorem 3.2. *If $q(x)$ is continuous and $q(x) > 0$ for all $x \geq A$ and if $\int_A^\infty q(x)dx = \infty$, then any solution to $y'' + qy = 0$ has infinitely many zeros for $x \geq A$.*

Proof. Suppose not. Then there is a solution y for which $y(x) > 0$ for all $x \geq B$, for some $B \geq A$. If we had $y'(C) \leq 0$ for some $C > B$, then as $y'' < 0$ and therefore y' is decreasing, we'd have that $y'(x) < 0$ for all $x > C$, and therefore there is some $x > C$ with $y(x) = 0$. So it must be that $y'(x) > 0$ for all $x \geq B$. Now consider $V(x) := -\frac{y'(x)}{y(x)}$. We already know it is negative for all $x \geq B$. Yet

$$V' = -\frac{y''y - y'^2}{y^2} = \frac{qy^2 + y'^2}{y^2} = q + V^2,$$

and hence

$$V(x) = V(B) + \int_B^x V'(t)dt = V(B) + \int_B^x V^2 dt + \int_B^x q dt.$$

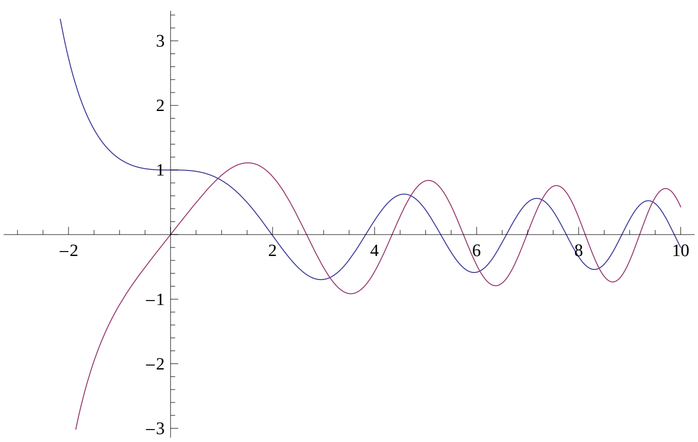
But as $\int_B^\infty q(t)dt$ is divergent, the above quantity will become positive for large enough x, contradicting the negativity of $V(x)$. □

Example 3.2. Solutions of Airy's equation $y'' + xy = 0$ oscillate for positive x but do not oscillate for negative x:

```
Ai1 = NDSolve[y''[x] + x y[x] == 0 && y[0] == 1 && y'[0] == 0,
   y[x], {x, -3, 10}];
Ai2 = NDSolve[y''[x] + x y[x] == 0 && y[0] == 0 && y'[0] == 1,
   y[x], {x, -3, 10}];
Ai = Join[Ai1, Ai2]

{{y[x] → InterpolatingFunction[{{-3., 10.}}, <>][x]},
 {y[x] → InterpolatingFunction[{{-3., 10.}}, <>][x]}}

Plot[Evaluate[y[x] /. Ai], {x, -3, 10}]
```



George Biddell Airy,
1801–1892

In the other direction, we have the following:

Theorem 3.3. *Let $A > 0$ be given. If $q(x)$ is continuous and $q(x) > 0$ for all $x \geq A$ and if $\int_A^\infty xq(x)dx < \infty$, and if y is a solution of $y'' + qy = 0$, then*

1. *There is some $B > A$ beyond which y has no zeros.*
2. *There is a constant K such that*

$$\lim_{x\to\infty} y'(x) = K = \lim_{x\to\infty} \frac{y(x)}{x}$$

Comment 3.3. I could not prove or find a counterexample to the statement that above, K is always non-zero. If this is true then the first statement above is superfluous as it would immediately follow from the second. I didn't have time to consult with the references, [CL, page 103, problem 28] and [Co, page 92 Theorem 3].

Proof. Find $C > A$ such that $\int_C^\infty xqdx < 1$, and assume that y has at least two zeros beyond C; let a be the first of those and let b be the second. Let $\alpha = y'(a)$; without loss of generality we may assume that $\alpha > 0$. Then $y'(b) < 0$ and by convexity we have that on $[a, b]$, $y(x) \leq \alpha(x - a) < \alpha x$. So

$$\alpha \leq y'(a) - y'(b) = -\int_a^b y''(x)dx = \int_a^b yqdx \leq \int_a^b \alpha xqdx \leq \alpha \int_C^\infty xqdx < \alpha,$$

a contradiction. Therefore y cannot have two further zeros beyond B, and (1) is proven.

Now we know that beyond some point D, y is non-zero. Without loss of generality it is positive and therefore convex. It therefore lies below any of its tangents, and therefore on $[D, \infty]$ it is bounded by some linear function βx. Hence for any $a < b$ in $[D, \infty]$,

$$|y'(a) - y'(b)| = \left|\int_a^b y''dx\right| = \int_a^b yqdx \leq \int_a^b \beta xqdx \leq \beta \int_a^\infty xqdx,$$

and the last integral goes to 0 when $a \to \infty$. Hence $y'(x)$ is a "Cauchy function" (the "function" analog of a "Cauchy sequence"), and hence it converges to some limit K. The rest follows from L'Hôpital. □

Exercise 3.2. Show that solutions of $y'' + (\log x)y = 0$ oscillate as $x \to \infty$, yet have at most one zero for $0 < x < 1$.

Exercise 3.3. Determine the behaviour of solutions of $y'' + \frac{x^2-2}{x^2(x^2+1)^2}y = 0$ as $x \to \infty$.

Exercise 3.4. What do the above theorems say about the behaviour of solutions of $y'' + \frac{y}{x^2} = 0$ near ∞? What is their actual behaviour?

Exercise 3.5. Show that all solutions of $y'' + x^\alpha y = 0$ are oscillatory for $x > 1$ if $\alpha > -1$. For what value of α does Theorem 3.3 apply to determine the large x behaviour of such solutions?

Exercise 3.6. Let y be the solution of

$$y'' + (x^2 - 1)^{1/3}y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Does $y(x)$ have other zeros for $-\infty < x < \infty$? Does it have infinitely many? What intervals $a < x < b$ cannot contain any other zeros?

Exercise 3.7. How do solutions of

$$y'' + \frac{1}{(t^2+1)^{3/2}} y = 0$$

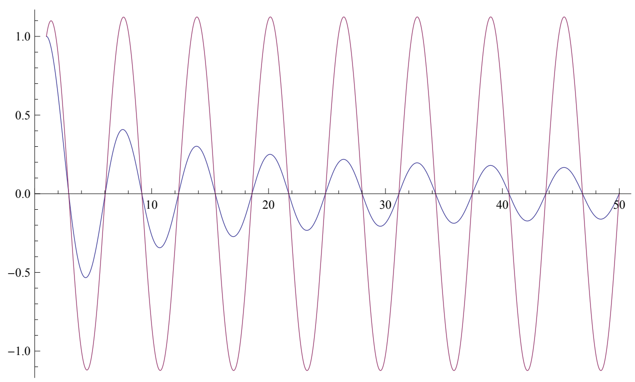
behave as $t \to \infty$? As $t \to -\infty$?

4. Changes of Variables

4.1. **Changing the Dependent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $y = \mu(x)V$, where μ satisfies $2\mu' + p\mu = 0$, then V satisfies $V'' + Q(x)V = 0$, where $Q = q - \frac{1}{4}p^2 - \frac{1}{2}p'$. The good news is that V has exactly the same zeros as y, so the "frequency" of the oscillatory behaviour of y may be studied by studying $V'' + Q(x)V = 0$. Though note that "amplitudes" are modified.

Example 4.1. For Bessel's equation of order 0, $y'' + \frac{1}{x}y' + y = 0$, which appeared here in Example 1.1, setting $V = \sqrt{x}y$ yields the equation $V'' + \left(1 + \frac{1}{4x^2}\right)V = 0$, which oscillates by Theorem 3.2:

```
{V₀} = NDSolve[
   V''[x] + (1 + 1/(4 x^2)) V[x] == 0
    && V[1] == 1 && V'[1] == 1 / 2,
   V[x], {x, 1, 50}
  ];
Plot[Evaluate[{y[x] /. J₀, V[x] /. V₀}], {x, 1, 50}]
```



4.2. **Changing the Independent Variable.** If y satisfies $y'' + p(x)y' + q(x)y = 0$ and we set $z = \nu(x)$, where ν satisfies $\nu'' + p\nu' = 0$, then the equation becomes

$$\frac{d^2y}{dz^2} + Q(z)y = 0, \qquad \text{where} \qquad Q(z) = \frac{q(x(z))}{[\nu'(x(z))]^2}.$$

The zeros of y get moved by this transformation, so studying the oscillatory behaviour of $y(x)$ as $x \to \infty$ corresponds to studying the oscillatory behaviour of $y(z)$ as $z \to \lim_{x\to\infty} \nu(x)$, and the latter point may or may not be ∞. Note though, that the amplitudes of oscillations (if they occur), are unchanged.

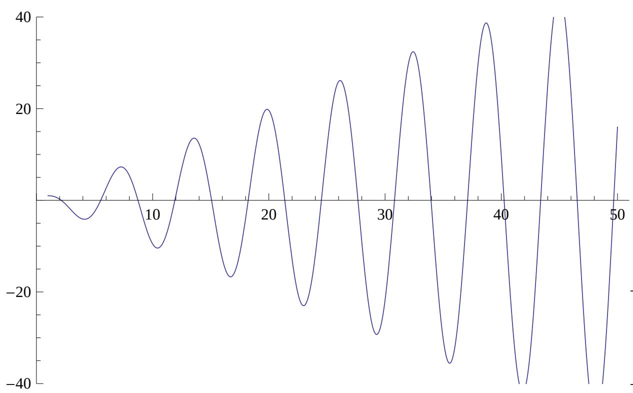
Exercise 4.1. Bring the Bessel equation of order 0 to the form $\frac{d^2y}{dz^2} + Q(z)y = 0$ by a change of the independent variable and verify once more that its solutions oscillate as $x \to \infty$.

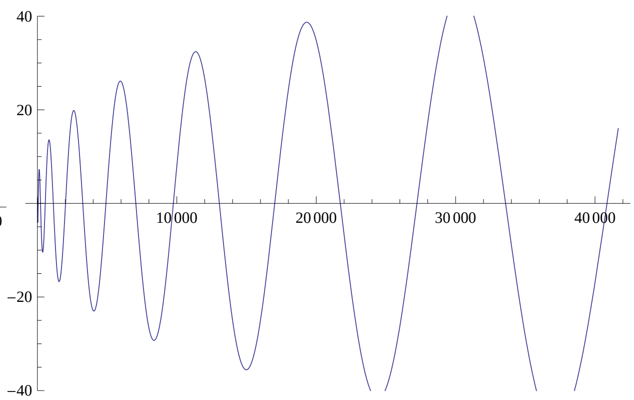
Exercise 4.2. Try to determine the behaviour of solutions of the equation $y'' + y'/x + y/x^3 = 0$ as $x \to \infty$, first by a change of the dependent variable and then by a change of the independent variable.

Example 4.2. Under the change of independent variable $z(x) = x^3/3$, the equation $y'' - \frac{2}{x}y' + y = 0$ becomes the equation $\frac{d^2y}{dz^2} + \frac{1}{(3z)^{4/3}}y = 0$:

```
a = 50; b = 40;
ψ = NDSolve[
   y''[x] - 2/x y'[x] + y[x] == 0
    && y[1] == 1 && y'[1] == 0,
   y[x], {x, 1, a}
  ];
Plot[Evaluate[y[x] /. ψ],
 {x, 1, a}, PlotRange → {-b, b}]
```

```
ψ = NDSolve[
   y''[z] + 1/(3 z)^(4/3) y[z] == 0
    && y[1/3] == 1 && y'[1/3] == 0,
   y[z], {z, 1, a^3/3}
  ];
Plot[Evaluate[y[z] /. ψ],
 {z, 1, a^3/3}, PlotRange → {-b, b}]
```





Exercise 4.3. For each of the following equations, decide whether their solutions oscillate for large x (here $n > 0$):

(1) $x^2y'' + xy' + y = 0$.
(2) $xy'' + (1 - x)y' + ny = 0$.
(3) $y'' - 2xy' + 2ny = 0$.
(4) $xy'' + (2n + 1)y' + xy = 0$.

Exercise 4.4. (1) Study whether solutions of $x^2y'' - xy' + 5y = 0$ oscillate as $x \to \infty$ and as $x \to -\infty$.
(2) Do the same for $x^2y'' - 4xy' + (6 - x)y = 0$.

Exercise 4.5. Are there any values of k for which solutions to $(1 - x)y'' - xy' + ky = 0$ oscillate as $x \to \infty$?

Exercise 4.6. How do solutions of

$$x(x-1)y'' + (3x - \frac{1}{2})y' + y = 0$$

behave as $x \to \infty$?

Exercise 4.7. How do solutions of

$$y'' + \frac{1}{x^2}y' + \frac{1}{4x^4}y = 0$$

behave as $x \to \infty$?

5. The Sturm Comparison Theorem

Charles Sturm, 1803–1855

Theorem 5.1. *(The Sturm Comparison Theorem) Suppose y_1 satisfies $y_1'' + q_1 y_1 = 0$ and y_2 satisfies $y_2'' + q_2 y_2 = 0$ and suppose $q_2 > q_1$ in some interval. Then in the open interval between any two zeros of y_1 there is a zero of y_2 (hence y_2 oscillates more rapidly than y_1).*

Proof. Consider $W(x) := y_1(x)y_2'(x) - y_2(x)y_1'(x)$. Then

$$W' = y_1 y_2'' - y_2 y_1'' = (q_1 - q_2) y_1 y_2.$$

Now argue by contradiction. Suppose a and b are successive zeros of y_1, and $a < b$, and that y_2 has no zeros on (a, b). On (a, b) the solution y_1 is non-zero; without loss of generality, it is positive. This implies that $y_1'(a) > 0$ and $y_1'(b) < 0$. Also without loss of generality, $y_2 > 0$ on (a, b). Then by the above equality and by $q_1 < q_2$, it follows that W is decreasing on (a, b). Yet $W(a) = -y_2(a)y_1'(a) \leq 0$ and $W(b) = -y_2(b)y_1'(b) \geq 0$. □

Corollary 5.1. Assuming $y'' + qy = 0$, if q is increasing the the distance between successive zeros of y is decreasing, and if q is decreasing then the distance between successive zeros of y is increasing.

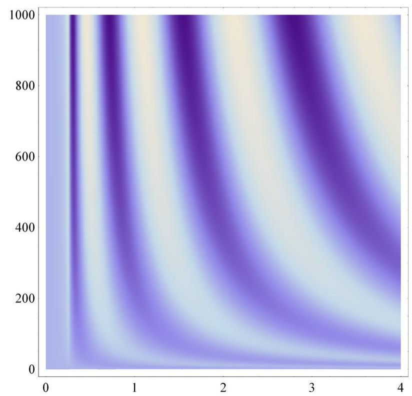
Proof. Assume for example that q is increasing, and that $a < b$ and $c < d$ are two pairs of successive zeros of y, with $c > a$. Then $y_1(x) := y(x + c - a)$ solves $y_1'' + q_1 y_1 = 0$, where $q_1(x) := q(x + c - a)$, and quite clearly, a and $d + a - c$ are successive zeros of y_1. But $q_1 > q$, and for y, the next zero after a is b, meaning that the next zero of y_1 must come before b. Namely, $d + a - c < b$, or alternatively, $d - c < b - a$, as required. □

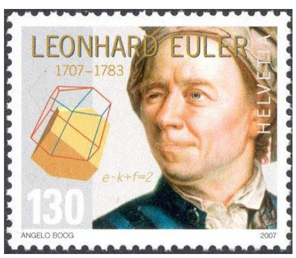
Example 5.2. As we have seen in Example 4.1 the Bessel equation of order 0 is equivalent to the equation $V'' + \left(1 + \frac{1}{4x^2}\right) V = 0$. Hence the distance between successive zeros of the Bessel equation of order 0 is increasing and by comparison with $v'' + v = 0$, it converges to π:

```
zs = x /. Table[FindRoot[y[x] /. J0, {x, λ}], {λ, 2.8, 50, 3.14}]
{2.91009, 6.03123, 9.16593, 12.3041, 15.4436, 18.5839, 21.7245, 24.8654,
 28.0064, 31.1475, 34.2888, 37.43, 40.5714, 43.7127, 46.8541, 49.9956}
Table[zs[[j + 1]] - zs[[j]], {j, 1, 15}]
{3.12114, 3.1347, 3.13816, 3.13954, 3.14023, 3.14062, 3.14087,
 3.14103, 3.14114, 3.14123, 3.14129, 3.14133, 3.14137, 3.1414, 3.14143}
```

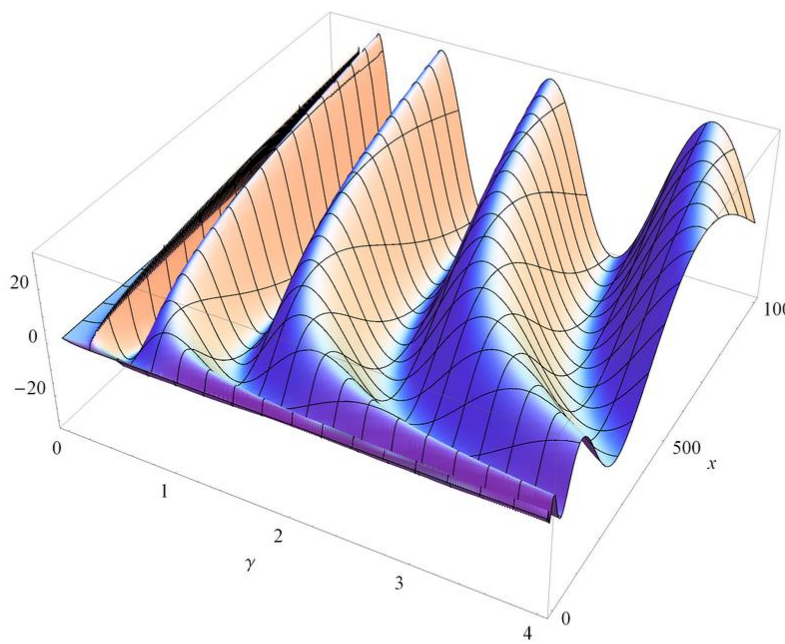
Example 5.3. Solutions of Euler's equation $x^2y'' + \gamma y = 0$ oscillate for $\gamma > \frac{1}{4}$ but do not oscillate for $\gamma \leq \frac{1}{4}$:

```
Rasterize[DensityPlot[
  If[γ < 1/4, x^((1-√(1-4γ))/2),
   √x Cos[Log[x] √(4γ-1)]],
  {γ, 0, 4}, {x, 0, 1000},
  PlotPoints → 100, LabelStyle → Medium
 ], ImageSize → 360]
```






```
Rasterize[Plot3D[
  If[γ < 1/4, x^((1-√(1-4γ))/2), √x Cos[Log[x] √(4γ-1)]],
  {γ, 0, 4}, {x, 0, 1000},
  PlotPoints → 100, AxesLabel → Automatic,
  LabelStyle → Medium
 ], ImageSize → 500]
```



Corollary 5.4. Suppose there exist numbers $\gamma > \frac{1}{4}$ and A such that for all $x \geq A$ we have $q(x) > \frac{\gamma}{x^2}$. Then every solution of $y'' + qy = 0$ oscillates infinitely often for $x > A$. However if for all $x \geq A$ we have $q(x) \leq \frac{\gamma}{4x^2}$, then solutions of $y'' + qy = 0$ have at most one zero for $x \geq A$.

Exercise 5.1. Construct an equation $y'' + qy = 0$ whose solutions oscillate, yet so slowly that even the above corollary would not detect these oscillations. [Note that any such equation can be used as a finer comparison criterion than the one in the corollary].

Hint. Change the independent variable to slow things down, and then the dependent variable to bring them back to the right form.

```
eq = x^2 y''[x] + γ y[x] /. y → (Y[e^#] &) /.
   x → Log[z];
eq = Expand[eq / Coefficient[eq, Y''[z]]]
```

$$\frac{\gamma Y[z]}{z^2 \text{Log}[z]^2} + \frac{Y'[z]}{z} + Y''[z]$$

```
{p, q} = Coefficient[eq, #] & /@
   {Y'[z], Y[z]};
Q = q - 1/4 p^2 - 1/2 ∂_z p
```

$$\frac{1}{4 z^2} + \frac{\gamma}{z^2 \text{Log}[z]^2}$$

Exercise 5.2. What can you say about the spacing of the zeros of the following equations:

(1) $y'' + (x^2 - 1)^{1/3} y = 0$.
(2) $y'' - (x - x^3)y = 0$.

Exercise 5.3. Let y be a solution of Bessel's equation of order α:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right) y = 0.$$

(1) Show that if $\alpha^2 < \frac{1}{4}$ then successive zeros of y are separated by less than π.
(2) Show that if $\alpha^2 > \frac{1}{4}$ then successive zeros of y are separated by more than π.
(3) What if $\alpha^2 = \frac{1}{4}$?

Exercise 5.4. Show that all solutions of $y'' + \left(\frac{1}{4x^2} + e^{-x}\right) y = 0$ do not oscillate.

Exercise 5.5. Study the $x \to \infty$ behaviour of solutions of $y'' + \frac{3}{x}y' + \left(\frac{1}{x^2} - \frac{1}{2x^4}\right) y = 0$.

Exercise 5.6. For which values of k to all solutions of $(x^2 - 1)y'' + xy' + ky = 0$ oscillate as $x \to \infty$?

Exercise 5.7. Prove that if $q(x) \to L > 0$ as $x \to \infty$, then the spacing between successive zeros of solutions of $y'' + qy = 0$ converges to $\frac{\pi}{\sqrt{L}}$ as $x \to \infty$.

Exercise 5.8. Prove the "Sturm Separation Theorem": If y_1 and y_2 are two linearly independent solutions of the same equation $y'' + p(x)y' + q(x)y = 0$, then their zeros alternate. Namely, between any two zeros of y_1 there is a zero of y_2 and between any two zeros of y_2 there is a zero of y_1.

6. Amplitudes

Theorem 6.1. *Consider a solution y of the equation $y'' + py' + qy = 0$. If $q > 0$ and $q' + 2pq > 0$ on some interval $[a, b]$ and $y'(a) = 0 = y'(b)$, then $|y(a)| > |y(b)|$. If instead $q' + 2pq < 0$ and $y'(a) = 0 = y'(b)$, then $|y(a)| < |y(b)|$. Similarly for non-strict inequalities.*

Proof. Consider $F = y^2 + \frac{(y')^2}{q}$ and note that $F' = -(q' + 2pq)\frac{(y')^2}{q^2}$. □

Example 6.1. For Bessel's equation $y'' + \frac{1}{x}y' + (1 - \alpha^2/x^2)y = 0$ we have $q' + 2pq = 2/x > 0$, and hence the amplitudes of its oscillations decreases on $x > 0$. Yet for $y'' + y/x^2 = 0$ we have $q' + 2pq = \frac{-2}{x^3} < 0$, and hence the amplitudes of its oscillations increases on $x > 0$.

Theorem 6.1 has the following "opposite" (really, strengthening):

Proposition 6.2. *Under the same conditions as in the theorem, let P be some primitive of p, meaning $P' = p$. Then*

$$e^{P(a)}\sqrt{q(a)}|y(a)| < e^{P(b)}\sqrt{q(b)}|y(b)| \qquad \text{if} \qquad q' + 2pq > 0,$$

and

$$e^{P(a)}\sqrt{q(a)}|y(a)| > e^{P(b)}\sqrt{q(b)}|y(b)| \qquad \text{if} \qquad q' + 2pq < 0.$$

Proof. Use the auxiliary function $G(x) = e^{2P}(qy^2 + (y')^2)$. □

Corollary 6.2. If $y'' + qy = 0$ where $q(x) \to L > 0$ monotonically as $x \to \infty$, then y oscillates as $x \to \infty$ with amplitudes that approach a finite, non-zero level.

Exercise 6.1. Describe, as best as you can at this stage, the behaviour as $x \to \infty$ of solutions of the equation $y'' + \left(1 - \frac{2}{x^2}\right) y = 0$.

Example 6.3. Under the transformation $v = \sqrt{x}y$ Bessel's equation $y''+\frac{1}{x}y+\left(1 - \frac{\alpha^2}{x^2}\right) y = 0$ becomes the equation

$$v'' + \left(1 + \frac{1-4\alpha^2}{4x^2}\right) v = 0.$$

Thus we see that the oscillations of v increase if $\alpha < \frac{1}{2}$ and decrease if $\alpha > \frac{1}{2}$. Further, they approach a constant level — but this means that the oscillations of y decrease like $\frac{1}{\sqrt{x}}$.

More can and should be said, though perhaps not on this handout.

7. Irregular Singular Points

Behaviour of solutions near a finite *irregular* singular point x_0 can sometimes be studied by the change of variables $t = 1/(x - x_0)$. More can and should be said, though perhaps not on this handout.

References

[CL] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* McGraw-Hill, New York 1955.

[Co] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations,* Heath, Boston 1965.

Dror Bar-Natan, December 21, 2012; `http://drorbn.net/index.php?title=12-267`.

Sources at `http://drorbn.net/AcademicPensieve/Classes/12-267/QualitativeAnalysis/`.

```
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  PlotStyle → {Pink, LightBlue}
 ], ImageSize → 750]
```

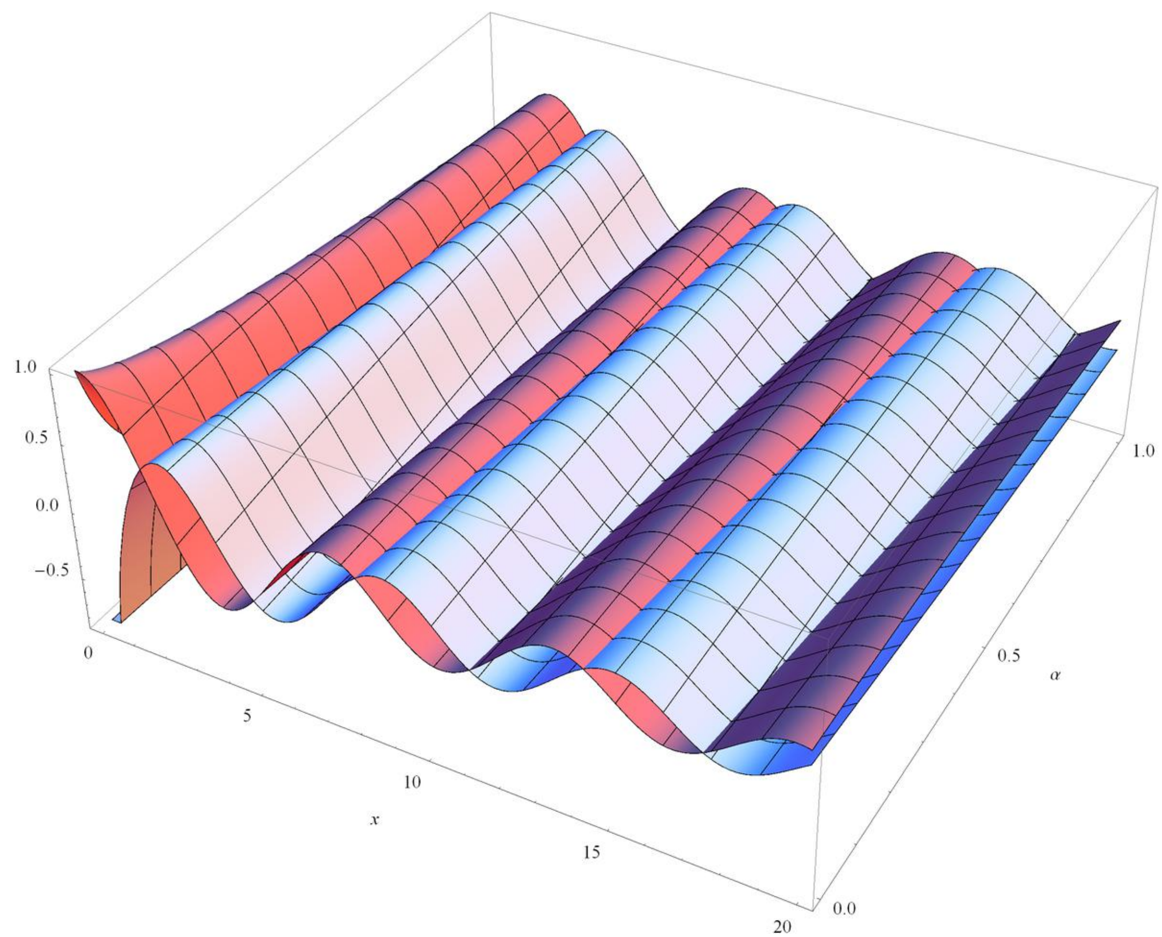

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$