The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues pensieve://2016-06/.

(Alt) In[•]:=
```
δ /: δ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
Otherwise = True;
```

The Burau Representation

In[•]:=
```
Inverse[{{0, t}, {1, 1 - t}}] /. t → t^-1 // Simplify // Transpose // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

In[•]:=
```
B_i_,j_[ξ_] := ξ /. v_j :> (1 - t) v_i + t v_j;
B̄_i_,j_[ξ_] := ξ /. v_k_ :> { (1 - t^-1) v_i + t^-1 v_j   j == k
                              v_k                          Otherwise ;
```

In[•]:=
```
{{v1, v2, v3} // B_1,3, {v1, v2, v3} // B̄_1,3}
```

Out[•]=
$$\left\{\{v_1, v_2, (1-t)\, v_1 + t\, v_3\}, \{v_1, v_2, \left(1 - \frac{1}{t}\right) v_1 + \frac{v_3}{t}\}\right\}$$

In[•]:=
```
{v1, v2, v3} // B_1,3 // B̄_1,3 // Expand
```

Out[•]=
$$\{v_1, v_2, v_3\}$$

In[•]:=
```
Column@{R3l = {v1, v2, v3} // B_1,2 // B_1,3 // B_2,3,
  R3r = {v1, v2, v3} // B_2,3 // B_1,3 // B_1,2 ,
  R3l - R3r // Expand}
```

Out[•]=
$$\{v_1, (1-t)\, v_1 + t\, v_2, (1-t)\, v_1 + t\, ((1-t)\, v_2 + t\, v_3)\}$$
$$\{v_1, (1-t)\, v_1 + t\, v_2, (1-t)\, ((1-t)\, v_1 + t\, v_2) + t\, ((1-t)\, v_1 + t\, v_3)\}$$
$$\{0, 0, 0\}$$

The Gassner Representation

In[•]:=
```
G_i_,j_[ξ_] := ξ /. v_k_ :> v_k + δ_k,j (t_i - 1) (v_j - v_i);
Ḡ_i_,j_[ξ_] := ξ /. v_j :> (1 - t_i^-1) v_i + t_i^-1 v_j;
```

In[•]:=
```
{v1, v2, v3} // G_1,3 // Ḡ_1,3 // Expand
```

Out[•]=
$$\{v_1, v_2, v_3\}$$

```
In[•]:= Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
          R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2,
          R3l - R3r // Expand}
```

$$\left\{v_1,\ v_2+\left(-1+t_1\right)\ \left(-v_1+v_2\right),\ v_3+\left(-1+t_2\right)\ \left(-v_2+v_3\right)+\left(-1+t_1\right)\ \left(-v_1+v_3+\left(-1+t_2\right)\ \left(-v_2+v_3\right)\right)\right\}$$

```
Out[•]= {v1, v2 + (-1 + t1) (-v1 + v2),
          v3 + (-1 + t1) (-v1 + v3) + (-1 + t2) (-v2 - (-1 + t1) (-v1 + v2) + v3 + (-1 + t1) (-v1 + v3))}
        {0, 0, 0}
```

```
In[•]:= Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
          OCr = {v1, v2, v3} // G1,3 // G1,2 ,
          OCl - OCr // Expand}
```

$$\left\{v_1,\ v_2+\left(-1+t_1\right)\ \left(-v_1+v_2\right),\ v_3+\left(-1+t_1\right)\ \left(-v_1+v_3\right)\right\}$$

```
Out[•]= {v1, v2 + (-1 + t1) (-v1 + v2), v3 + (-1 + t1) (-v1 + v3)}
        {0, 0, 0}
```

```
In[•]:= Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
          UCr = {v1, v2, v3} // G2,3 // G1,3 ,
          UCl - UCr // Expand}
```

$$\left\{v_1,\ v_2,\ v_3+\left(-1+t_2\right)\ \left(-v_2+v_3\right)+\left(-1+t_1\right)\ \left(-v_1+v_3+\left(-1+t_2\right)\ \left(-v_2+v_3\right)\right)\right\}$$

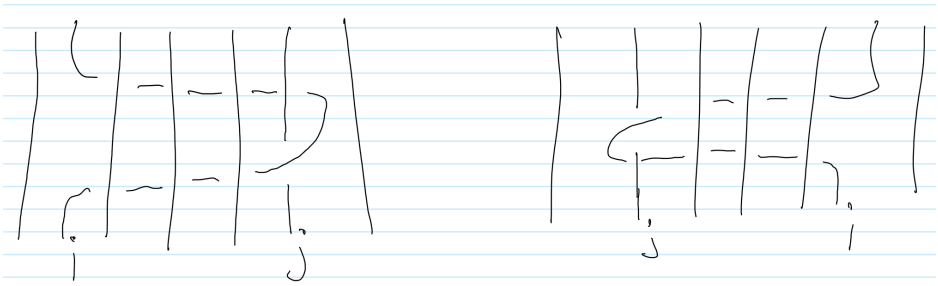
```
Out[•]= {v1, v2, v3 + (-1 + t1) (-v1 + v3) + (-1 + t2) (-v2 + v3 + (-1 + t1) (-v1 + v3))}
        {0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

The Gassner Representation for u-Braids

```
(Alt) In[•]:= γi_[ℰ_] := Expand[ℰ /. vi+1 :> (1 - ti) vi + ti vi+1 /. {ti → ti+1, ti+1 → ti, vi → vi+1, vi+1 → vi}];
              γ̄i_[ℰ_] := Expand[ℰ /. {ti → ti+1, ti+1 → ti, vi → vi+1, vi+1 → vi} /. vi+1 :> (1 - ti^-1) vi + ti^-1 vi+1];

(Alt) In[•]:= {v1, v2, v3} // γ1 // γ̄1

(Alt) Out[•]= {v1, v2, v3}
```



(Alt) In[•]:=
```
γ_{i_,j_}[ξ_] /; i < j := Module[{ξ = ξ},
   Do[ξ = γ̄_k[ξ], {k, i, j - 2}];
   ξ = ξ // γ_{j-1};
   Do[ξ = γ_k[ξ], {k, j - 1, i, -1}];
   ξ];
γ_{i_,j_}[ξ_] /; j < i := Module[{ξ = ξ},
  Do[ξ = γ_k[ξ], {k, i - 1, j, -1}];
  ξ = ξ // γ_j;
  Do[ξ = γ̄_k[ξ], {k, j + 1, i - 1}];
  ξ]
```

(Alt) In[•]:=
```
lhs = Table[v_i, {i, 7}] // γ_{2,6} // γ_{3,5}
```

(Alt) Out[•]=
$$\left\{v_1,\ t_6 v_2 + v_3 - \frac{v_3}{t_3} - t_6 v_3 + \frac{t_6 v_3}{t_3} + \frac{v_4}{t_3} - \frac{v_4}{t_3 t_4} - \frac{t_6 v_4}{t_3} + \frac{t_6 v_4}{t_3 t_4} + \frac{v_5}{t_3 t_4} - \frac{v_5}{t_3 t_4 t_5} - \frac{t_6 v_5}{t_3 t_4} + \frac{t_6 v_5}{t_3 t_4 t_5} + \frac{v_6}{t_3 t_4 t_5} - \frac{t_6 v_6}{t_3 t_4 t_5},\ t_5 v_3 + v_4 - \frac{v_4}{t_4} - t_5 v_4 + \frac{t_5 v_4}{t_4} + \frac{v_5}{t_4} - \frac{t_5 v_5}{t_4},\right.$$
$$v_4,\ t_4 t_5 v_3 - t_3 t_4 t_5 v_3 + t_5 v_4 - t_3 t_5 v_4 - t_4 t_5 v_4 + t_3 t_4 t_5 v_4 + v_5 - t_5 v_5 + t_3 t_5 v_5,$$
$$t_3 t_4 t_5 t_6 v_2 - t_2 t_3 t_4 t_5 t_6 v_2 + t_4 t_5 t_6 v_3 - t_2 t_4 t_5 t_6 v_3 - t_3 t_4 t_5 t_6 v_3 + t_2 t_3 t_4 t_5 t_6 v_3 + t_5 t_6 v_4 -$$
$$\left. t_2 t_5 t_6 v_4 - t_4 t_5 t_6 v_4 + t_2 t_4 t_5 t_6 v_4 + t_6 v_5 - t_2 t_6 v_5 - t_5 t_6 v_5 + t_2 t_5 t_6 v_5 + v_6 - t_6 v_6 + t_2 t_6 v_6,\ v_7\right\}$$

(Alt) In[•]:=
```
rhs = Table[v_i, {i, 7}] // γ_{3,5} // γ_{2,6}; lhs - rhs
```

(Alt) Out[•]=
$$\{0, 0, 0, 0, 0, 0, 0\}$$

(Alt) In[•]:=
```
Table[Simplify@Coefficient[γ_{2,4}[v_j], v_i], {i, 5}, {j, 5}] // MatrixForm
```

(Alt) Out[•]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t_4 & 0 & -\left(-1+t_2\right) t_3 t_4 & 0 \\ 0 & -\frac{(-1+t_3)(-1+t_4)}{t_3} & 1 & \left(-1+t_2\right)\left(-1+t_3\right) t_4 & 0 \\ 0 & \frac{1-t_4}{t_3} & 0 & 1+\left(-1+t_2\right) t_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Alt) In[•]:=
```
Table[Simplify@Coefficient[γ_{4,2}[v_j], v_i], {i, 5}, {j, 5}] // MatrixForm
```

(Alt) Out[•]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t_4 & 0 & -\left(-1+t_2\right) t_3 t_4 & 0 \\ 0 & -\frac{(-1+t_3)(-1+t_4)}{t_3} & 1 & \left(-1+t_2\right)\left(-1+t_3\right) t_4 & 0 \\ 0 & \frac{1-t_4}{t_3} & 0 & 1+\left(-1+t_2\right) t_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Alt) In[•]:=
```
Table[Simplify@∂_{v_i} γ_{2,5}[v_j], {i, 6}, {j, 6}] // MatrixForm
```

(Alt) Out[•]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_5 & 0 & 0 & -\left(-1+t_2\right) t_3 t_4 t_5 & 0 \\ 0 & -\frac{(-1+t_3)(-1+t_5)}{t_3} & 1 & 0 & \left(-1+t_2\right)\left(-1+t_3\right) t_4 t_5 & 0 \\ 0 & -\frac{(-1+t_4)(-1+t_5)}{t_3 t_4} & 0 & 1 & \left(-1+t_2\right)\left(-1+t_4\right) t_5 & 0 \\ 0 & \frac{1-t_5}{t_3 t_4} & 0 & 0 & 1+\left(-1+t_2\right) t_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Alt) In[•]:=
```
Table[Simplify@∂_{v_i} γ_{2,7}[v_j], {i, 8}, {j, 8}] // MatrixForm
```

(Alt) Out[•]//MatrixForm=

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_7 & 0 & 0 & 0 & 0 & -(-1+t_2)\,t_3\,t_4\,t_5\,t_6\,t_7 & 0 \\
0 & -\frac{(-1+t_3)\,(-1+t_7)}{t_3} & 1 & 0 & 0 & 0 & (-1+t_2)\,(-1+t_3)\,t_4\,t_5\,t_6\,t_7 & 0 \\
0 & -\frac{(-1+t_4)\,(-1+t_7)}{t_3\,t_4} & 0 & 1 & 0 & 0 & (-1+t_2)\,(-1+t_4)\,t_5\,t_6\,t_7 & 0 \\
0 & -\frac{(-1+t_5)\,(-1+t_7)}{t_3\,t_4\,t_5} & 0 & 0 & 1 & 0 & (-1+t_2)\,(-1+t_5)\,t_6\,t_7 & 0 \\
0 & -\frac{(-1+t_6)\,(-1+t_7)}{t_3\,t_4\,t_5\,t_6} & 0 & 0 & 0 & 1 & (-1+t_2)\,(-1+t_6)\,t_7 & 0 \\
0 & \frac{1-t_7}{t_3\,t_4\,t_5\,t_6} & 0 & 0 & 0 & 0 & 1+(-1+t_2)\,t_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The Gassner-Plus Representation

In[•]:=
```
GP_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j :> (1 - t_i) u_i + t_i u_j,
      f_. v_j :> f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i + f t_i u_i }];
GP̄_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j :> (1 - t_i^-1) u_i + t_i^-1 u_j,
      f_. v_j :> f (1 - t_i^-1) v_i + f t_i^-1 v_j + (t_i^-1 - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i - f t_i^-1 u_i }];
```

In[•]:=
```
GPchecks = {f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3};
GPchecks // GP_{1,3} // GP̄_{1,3}
```

Out[•]=
```
{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3}
```

In[•]:=
```
R3l = GPchecks // GP_{1,2} // GP_{1,3} // GP_{2,3}
```

Out[•]=
```
{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 -
   f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 + t_2 u_1 f^(0,1,0)[t_1, t_2, t_3] -
   t_1 t_2 u_1 f^(0,1,0)[t_1, t_2, t_3] - t_1 u_1 f^(1,0,0)[t_1, t_2, t_3] + t_1^2 u_1 f^(1,0,0)[t_1, t_2, t_3],
  f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] t_1 t_2 u_2 + f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 +
   f[t_1, t_2, t_3] t_1 v_2 - f[t_1, t_2, t_3] t_1 t_2 v_2 + f[t_1, t_2, t_3] t_1 t_2 v_3 + t_3 u_1 f^(0,0,1)[t_1, t_2, t_3] -
   t_1 t_3 u_1 f^(0,0,1)[t_1, t_2, t_3] + t_1 t_3 u_2 f^(0,0,1)[t_1, t_2, t_3] - t_1 t_2 t_3 u_2 f^(0,0,1)[t_1, t_2, t_3] -
   t_1 t_2 u_2 f^(0,1,0)[t_1, t_2, t_3] + t_1 t_2^2 u_2 f^(0,1,0)[t_1, t_2, t_3] - t_1 u_1 f^(1,0,0)[t_1, t_2, t_3] +
   t_1^2 u_1 f^(1,0,0)[t_1, t_2, t_3], u_1, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_2 - t_1 t_2 u_2 + t_1 t_2 u_3}
```

In[•]:=
```
R3r = GPchecks // GP_{2,3} // GP_{1,3} // GP_{1,2}; R3l - R3r
```

Out[•]=
```
{0, 0, 0, 0, 0, 0}
```

In[•]:=
```
OCl = GPchecks // GP_{1,2} // GP_{1,3}
```

Out[•]=
```
{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 -
   f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 + t_2 u_1 f^(0,1,0)[t_1, t_2, t_3] -
   t_1 t_2 u_1 f^(0,1,0)[t_1, t_2, t_3] - t_1 u_1 f^(1,0,0)[t_1, t_2, t_3] + t_1^2 u_1 f^(1,0,0)[t_1, t_2, t_3],
  f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_3 +
   t_3 u_1 f^(0,0,1)[t_1, t_2, t_3] - t_1 t_3 u_1 f^(0,0,1)[t_1, t_2, t_3] - t_1 u_1 f^(1,0,0)[t_1, t_2, t_3] +
   t_1^2 u_1 f^(1,0,0)[t_1, t_2, t_3], u_1, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_3}
```

In[•]:=
```
OCr = GPchecks // GP_{1,3} // GP_{1,2}; OCl - OCr
```

Out[•]=
```
{0, 0, 0, 0, 0, 0}
```

Question. Does GP factor through G? How?

The End(G) Representation

```
In[•]:= EG_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j → (1 - t_i) u_i + t_i u_j, w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
        EGbar_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j → (1 - t_i^-1) u_i + t_i^-1 u_j, w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
        EGchecks = Flatten@Table[u_i w_j, {i, 3}, {j, 3}];
        EGchecks // EG_{1,3} // EGbar_{1,3}
```

$$\{u_1 w_1,\ u_1 w_2,\ u_1 w_3,\ u_2 w_1,\ u_2 w_2,\ u_2 w_3,\ u_3 w_1,\ u_3 w_2,\ u_3 w_3\}$$

```
In[•]:= Short[R3l = EGchecks // EG_{1,2} // EG_{1,3} // EG_{2,3}, 10]
```

$$\Big\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},\ \frac{u_1 w_3}{t_1 t_2},\ \ll 1 \gg,$$

$$\ll 1 \gg,\ \ll 1 \gg,\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} -$$

$$t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 + 2 u_1 w_3 -$$

$$\frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$

$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$

$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
In[•]:= R3r = EGchecks // EG_{2,3} // EG_{1,3} // EG_{1,2}; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
In[•]:= (# → Collect[EG_{i,j}[#], u_ w_, Simplify]) & /@ {u_k w_j, u_k w_i, u_j w_k, u_j w_i}
```

$$\Big\{u_k w_j \to \frac{u_k w_j}{t_i},\ u_k w_i \to u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j,\ u_j w_k \to (1 - t_i)\, u_i w_k + t_i u_j w_k,$$

$$u_j w_i \to (1 - t_i)\, u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i)\, u_j w_j\Big\}$$

The End(G)+c Representation

Is there topology behind this representation?

```
In[•]:= EGc_{i_,j_}[ξ_] := Expand[ξ /. {
          c_i → c_i - (1 - t_i^-1) u_i w_j, c_j → c_j + (1 - t_i^-1) u_i w_j,
          u_j → (1 - t_i) u_i + t_i u_j,
          w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
        EGc-bar_{i_,j_}[ξ_] := Expand[ξ /. {
          c_i → c_i + (t_i - 1) u_i w_j, c_j → c_j + (1 - t_i) u_i w_j,
          u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
          w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
        EGcchecks = {c1, c2, c3, u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3};
        EGcchecks // EGc_{1,3} // EGc-bar_{1,3}
Out[•]= {c1, c2, c3, u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3}

In[•]:= (# → Collect[EGc_{1,2}[#], u_ w_, Simplify]) & /@ EGcchecks
Out[•]= {c1 → c1 + (-1 + 1/t1) u1 w2, c2 → c2 + (1 - 1/t1) u1 w2,
         c3 → c3, u1 w1 → u1 w1 + (1 - 1/t1) u1 w2, u1 w2 → u1 w2/t1, u1 w3 → u1 w3,
         u2 w1 → (1 - t1) u1 w1 + t1 u2 w1 - (-1 + t1)^2 u1 w2/t1 + (-1 + t1) u2 w2, u2 w2 → (-1 + 1/t1) u1 w2 + u2 w2,
         u2 w3 → (1 - t1) u1 w3 + t1 u2 w3, u3 w1 → u3 w1 + (1 - 1/t1) u3 w2, u3 w2 → u3 w2/t1, u3 w3 → u3 w3}

In[•]:= u1 w2 + c2 // EGc_{1,2}
Out[•]= c2 + u1 w2

In[•]:= (# → Simplify[EGc_{1,2}[#] /. {u_i_ w_i_ → 1, u_i_ w_j_ /; i ≠ j :→ 0}]) & /@ EGcchecks
Out[•]= {c1 → c1, c2 → c2, c3 → c3, u1 w1 → 1, u1 w2 → 0, u1 w3 → 0,
         u2 w1 → 0, u2 w2 → 1, u2 w3 → 0, u3 w1 → 0, u3 w2 → 0, u3 w3 → 1}

In[•]:= Short[R3l = EGcchecks // EGc_{1,2} // EGc_{1,3} // EGc_{2,3}, 10]
Out[•]//Short= {c1 - u1 w2 + u1 w2/t1 - u1 w3 + u1 w3/t1, c2 + u1 w2 - u1 w2/t1 + u1 w3 - u1 w3/t1 - u1 w3/t2 + u1 w3/(t1 t2) - u2 w3 + u2 w3/t2,
         c3 + u1 w3/t2 - u1 w3/(t1 t2) + u2 w3 - u2 w3/t2, <<7>>, -u1 w2 + u1 w2/t1 + u2 w2 - t2 u2 w2 + t2 u3 w2 - u1 w3 + u1 w3/t1 +
          u1 w3/t2 - u1 w3/(t1 t2) + 2 u2 w3 - u2 w3/t2 - t2 u2 w3 - u3 w3 + t2 u3 w3, -u1 w3/t2 + u1 w3/(t1 t2) - u2 w3 + u2 w3/t2 + u3 w3}

In[•]:= R3r = EGcchecks // EGc_{2,3} // EGc_{1,3} // EGc_{1,2}; R3l - R3r
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The Turbo-Gassner Representation

```
In[•]:= TG_{i_,j_}[ξ_] := Expand[ξ /. {
          f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
             (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
             δ_{k,i} f (u_j - u_i) u_i w_j],
          u_j → (1 - t_i) u_i + t_i u_j,
          w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
        TG̅_{i_,j_}[ξ_] := Expand[ξ /. {
          f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i^-1) v_i + t_i^-1 v_j,
             (1 - t_i) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) (u_k /. u_j → (1 - t_i^-1) u_i + t_i^-1 u_j) u_i w_j,
             δ_{k,i} f (u_i - u_j) u_i w_j],
          u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
          w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
        TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};
        TGchecks // TG_{1,3} // TG̅_{1,3}

Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3}

In[•]:= Short[R3l = TGchecks // TG_{1,2} // TG_{1,3} // TG_{2,3}, 10]

Out[•]//Short= {<<1>>}

In[•]:= R3r = TGchecks // TG_{2,3} // TG_{1,3} // TG_{1,2}; R3l - R3r

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Short[OCl = TGchecks // TG_{1,2} // TG_{1,3}]

Out[•]//Short= {<<18>> + t1 u1^2 w3 f^(1,0,0)[t1, t2, t3], <<1>>, <<5>>, <<1>>/<<1>>, w3/t1}

In[•]:= OCr = TGchecks // TG_{1,3} // TG_{1,2}; OCl - OCr

Out[•]= {0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3,
  -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 - f[t1, t2, t3] t1 u1 u3 w2,
  0, 0, 0, 0, 0, 0}
```

A Finite-Rank Turbo-Gassner Representation

```
In[•]:= FTG_{i_,j_}[ℰ_] := Expand[ℰ /. {
          v_k_ :↦ v_k + δ_{k,j} ((t_i - 1) (v_j - v_i) + v_{i,j} - v_{i,i}) + δ_{k,i} (u_j - u_i) u_i w_j,
          v_{l_,k_} :↦
           v_{l,k} + (t_i - 1) (δ_{k,j} (v_{l,j} - v_{l,i}) + (δ_{l,i} - δ_{l,j} t_i^{-1} t_j) (u_k + δ_{k,j} (t_i - 1) (u_j - u_i)) u_i w_j),
          u_k_ :↦ u_k + δ_{k,j} (t_i - 1) (u_j - u_i),
          w_k_ :↦ w_k + (δ_{k,j} - δ_{k,i}) (t_i^{-1} - 1) w_j}];
        FTGbar_{i_,j_}[ℰ_] := Expand[ℰ /. {
          v_k_ :↦ v_k + δ_{k,j} t_i^{-2} (t_i (t_i - 1) (v_i - v_j) + v_{i,i} - v_{i,j}) + δ_{k,i} (u_i - u_j) u_i w_j,
          v_{l_,k_} :↦ v_{l,k} + δ_{k,j} (1 - t_i^{-1}) (v_{l,i} - v_{l,j}) +
             (1 - t_i) (t_i δ_{l,i} - t_j δ_{l,j}) (u_k + δ_{k,j} (t_i^{-1} - 1) (u_j - u_i)) u_i w_j,
          u_k_ :↦ u_k + δ_{k,j} (t_i^{-1} - 1) (u_j - u_i),
          w_k_ :↦ w_k + (δ_{k,j} - δ_{k,i}) (t_i - 1) w_j}];
        FTGchecks[n_] :=
          {Table[v_k, {k, n}], Table[v_{1,k}, {l, n}, {k, n}], Table[u_k, {k, n}], Table[w_k, {k, n}]};
        (FTGchecks[3] // FTG_{1,3} // FTGbar_{1,3}) - FTGchecks[3]
        Short[R3l = FTGchecks[3] // FTG_{1,2} // FTG_{1,3} // FTG_{2,3}, 10]
        R3r = FTGchecks[3] // FTG_{2,3} // FTG_{1,3} // FTG_{1,2}; R3l - R3r

Out[•]= {{0, 0, 0}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {0, 0, 0}, {0, 0, 0}}

Out[•]//Short= {{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,
    v_1 - t_1 v_1 + t_1 v_2 - u_1 u_2 w_3 + t_1 u_1 u_2 w_3 - t_1 u_2^2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3 + t_1 u_2 u_3 w_3 - v_{1,1} + v_{1,2},
    v_1 - t_1 v_1 + t_1 v_2 - t_1 t_2 v_2 + t_1 t_2 v_3 - v_{1,1} + v_{1,2} - t_2 v_{1,2} + t_2 v_{1,3} - t_1 v_{2,2} + t_1 v_{2,3}},
   {<<1>>}, {u_1, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_2 - t_1 t_2 u_2 + t_1 t_2 u_3},
   {w_1 + w_2 - w_2/t_1 + w_3 - w_3/t_1, w_2/t_1 + w_3/t_1 - w_3/(t_1 t_2), w_3/(t_1 t_2)}}

Out[•]= {{0, 0, 0}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {0, 0, 0}, {0, 0, 0}}

In[•]:= FTGchecks[3] // FTG_{1,2} // Column

{v_1 - u_1^2 w_2 + u_1 u_2 w_2, v_1 - t_1 v_1 + t_1 v_2 - v_{1,1} + v_{1,2}, v_3}
{{-u_1^2 w_2 + t_1 u_1^2 w_2 + v_{1,1}, -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 + t_1^2 u_1 u_2 w_2 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},
  -u_1 u_3 w_2 + t_1 u_1 u_3 w_2 + v_{1,3}}, {-t_2 u_1^2 w_2 + (t_2 u_1^2 w_2)/t_1 + v_{2,1},
  -2 t_2 u_1^2 w_2 + (t_2 u_1^2 w_2)/t_1 + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},
  -t_2 u_1 u_3 w_2 + (t_2 u_1 u_3 w_2)/t_1 + v_{2,3}}, {v_{3,1}, v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2}, v_{3,3}}}
{u_1, u_1 - t_1 u_1 + t_1 u_2, u_3}
{w_1 + w_2 - w_2/t_1, w_2/t_1, w_3}
```

In[•]:=
```
Short[OCl = FTGchecks[3] // FTG1,2 // FTG1,3, 100]
```

Out[•]//Short=

$$\Big\{\Big\{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,\ v_1 - t_1 v_1 + t_1 v_2 - u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3 - v_{1,1} + v_{1,2},\ v_1 - t_1 v_1 + t_1 v_3 - v_{1,1} + v_{1,3}\Big\},$$
$$\Big\{\Big\{-u_1^2 w_2 + t_1 u_1^2 w_2 - u_1^2 w_3 + t_1 u_1^2 w_3 + v_{1,1},\ -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 + t_1^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_2 w_3 + t_1^2 u_1 u_2 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},$$
$$-u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_3 w_2 + t_1^2 u_1 u_3 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_3 w_3 + t_1^2 u_1 u_3 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,3}\Big\},$$
$$\Big\{-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_{2,1},\ -2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_3 w_2 - t_1 t_2 u_1 u_3 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,3}\Big\},$$
$$\Big\{-t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + v_{3,1},\ -2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_2 w_3 - t_1 t_3 u_1 u_2 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2},$$
$$-2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_3 w_3 - t_1 t_3 u_1 u_3 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,3}\Big\}\Big\},$$
$$\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_1 - t_1 u_1 + t_1 u_3\},\ \Big\{w_1 + w_2 - \frac{w_2}{t_1} + w_3 - \frac{w_3}{t_1},\ \frac{w_2}{t_1},\ \frac{w_3}{t_1}\Big\}\Big\}$$

In[•]:=
```
Short[OCr = FTGchecks[3] // FTG1,3 // FTG1,2, 100]
```

Out[•]//Short=

$$\Big\{\Big\{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,\ v_1 - t_1 v_1 + t_1 v_2 - v_{1,1} + v_{1,2},\ v_1 - t_1 v_1 + t_1 v_3 + u_1 u_2 w_2 - t_1 u_1 u_2 w_2 - u_1 u_3 w_2 + t_1 u_1 u_3 w_2 - v_{1,1} + v_{1,3}\Big\},$$
$$\Big\{\Big\{-u_1^2 w_2 + t_1 u_1^2 w_2 - u_1^2 w_3 + t_1 u_1^2 w_3 + v_{1,1},\ -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 + t_1^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_2 w_3 + t_1^2 u_1 u_2 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},$$
$$-u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_3 w_2 + t_1^2 u_1 u_3 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_3 w_3 + t_1^2 u_1 u_3 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,3}\Big\},$$
$$\Big\{-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_{2,1},\ -2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_3 w_2 - t_1 t_2 u_1 u_3 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,3}\Big\},$$
$$\Big\{-t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + v_{3,1},\ -2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_2 w_3 - t_1 t_3 u_1 u_2 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2},$$
$$-2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_3 w_3 - t_1 t_3 u_1 u_3 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,3}\Big\}\Big\},$$
$$\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_1 - t_1 u_1 + t_1 u_3\},\ \Big\{w_1 + w_2 - \frac{w_2}{t_1} + w_3 - \frac{w_3}{t_1},\ \frac{w_2}{t_1},\ \frac{w_3}{t_1}\Big\}\Big\}$$

In[•]:=
```
OCl - OCr
```

Out[•]=
$$\{\{0,\ -u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3,\ -u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_3 w_2 - t_1 u_1 u_3 w_2\},\ \{\{0,0,0\},\{0,0,0\},\{0,0,0\}\},\ \{0,0,0\},\ \{0,0,0\}\}$$

In[•]:= **Short[FOC1 = FTGchecks[4] // FTG$_{1,2}$ // FTG$_{1,3}$ // FTG$_{1,4}$]**

Out[•]//Short= {{ <<1>> }, { <<1>> }, <<1>>, <<1>> }

In[•]:= **Short[FOC2 = FTGchecks[4] // FTG$_{1,3}$ // FTG$_{1,2}$ // FTG$_{1,4}$]**

Out[•]//Short= {{ <<1>> }, { <<1>> }, <<1>>, <<1>> }

In[•]:= **Short[FOC3 = FTGchecks[4] // FTG$_{1,4}$ // FTG$_{1,2}$ // FTG$_{1,3}$]**

Out[•]//Short= {{ <<1>> }, { <<1>> }, <<1>>, <<1>> }

In[•]:= **Short[FOC4 = FTGchecks[4] // FTG$_{1,4}$ // FTG$_{1,3}$ // FTG$_{1,2}$]**

Out[•]//Short= {{ <<1>> }, { <<1>> }, <<1>>, <<1>> }

In[•]:= **FOC1 - FOC2 - FOC3 + FOC4**

Out[•]= {{0, 0, 0, 0}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{0, 0, 0, 0}, {0, 0, 0, 0}}

In[•]:= **FOC1 - FOC2**

Out[•]= $\{\{0, -u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3, -u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_3 w_2 - t_1 u_1 u_3 w_2, 0\},$
$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[•]:= **FOC2 - FOC3**

Out[•]= $\{\{0, u_1 u_2 w_3 - t_1 u_1 u_2 w_3 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 - u_1 u_2 w_4 + t_1 u_1 u_2 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$u_1 u_2 w_2 - t_1 u_1 u_2 w_2 - u_1 u_3 w_2 + t_1 u_1 u_3 w_2 - u_1 u_3 w_4 + t_1 u_1 u_3 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$-u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_4 w_2 - t_1 u_1 u_4 w_2 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 + u_1 u_4 w_3 - t_1 u_1 u_4 w_3\},$
$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[•]:= **FOC1 - FOC3**

Out[•]= $\{\{0, -u_1 u_2 w_4 + t_1 u_1 u_2 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4, -u_1 u_3 w_4 + t_1 u_1 u_3 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$-u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_4 w_2 - t_1 u_1 u_4 w_2 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 + u_1 u_4 w_3 - t_1 u_1 u_4 w_3\},$
$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[•]:= **(FTGchecks[4] // FTG$_{1,2}$ // FTG$_{1,3}$ // $\overline{\text{FTG}}_{1,2}$ // $\overline{\text{FTG}}_{1,3}$ // FTG$_{1,4}$ // FTG$_{1,3}$ // FTG$_{1,2}$ // $\overline{\text{FTG}}_{1,3}$ //
$\overline{\text{FTG}}_{1,2}$ // $\overline{\text{FTG}}_{1,4}$) - FTGchecks[4]**

Out[•]= {{0, 0, 0, 0}, {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{0, 0, 0, 0}, {0, 0, 0, 0}}

In[•]:= **(FTGchecks[4] // FTG$_{1,2}$ // FTG$_{1,3}$ // $\overline{\text{FTG}}_{1,2}$ // $\overline{\text{FTG}}_{1,3}$ // FTG$_{1,3}$ // FTG$_{1,3}$ // FTG$_{1,2}$ // $\overline{\text{FTG}}_{1,3}$ //
$\overline{\text{FTG}}_{1,2}$ // $\overline{\text{FTG}}_{1,3}$) - FTGchecks[4]**

Out[•]= $\{\{0, u_1^2 w_3 - 2 t_1 u_1^2 w_3 + t_1^2 u_1^2 w_3 - u_1 u_2 w_3 + 2 t_1 u_1 u_2 w_3 - t_1^2 u_1 u_2 w_3,$
$u_1^2 w_2 - 2 t_1 u_1^2 w_2 + t_1^2 u_1^2 w_2 - u_1 u_2 w_2 + 2 t_1 u_1 u_2 w_2 - t_1^2 u_1 u_2 w_2, 0\},$
$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

```
In[•]:= (FTGchecks[5] // FTG1,2 // FTG1,3 // FTG̅1,2 // FTG̅1,3 // FTG1,4 // FTG1,5 // FTG̅1,4 // FTG̅1,5) -
         FTGchecks[5]

Out[•]= {{0, -u1 u2 w3 + t1 u1 u2 w3 + u1 u3 w3 - t1 u1 u3 w3, -u1 u2 w2 + t1 u1 u2 w2 + u1 u3 w2 - t1 u1 u3 w2,
          -u1 u4 w5 + t1 u1 u4 w5 + u1 u5 w5 - t1 u1 u5 w5, -u1 u4 w4 + t1 u1 u4 w4 + u1 u5 w4 - t1 u1 u5 w4},
         {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
         {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

The Turbo-Burau Representation

```
In[•]:= η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
        TBi_,j_[ξ_] :=
          Expand[ξ /. {
              f_. vk_ :> Plus[f vk /. vj → (1 - t - η[i]) vi + (t + η[i]) vj,
                 (t - 1) (∂η[i] f - ∂η[j] f) (uk /. uj → (1 - t) ui + t uj) ui wj,
                 δk,i (f /. _η → 0) (uj - ui) ui wj],
              uj → (1 - t) ui + t uj,
              wi → wi + (1 - t^-1) wj, wj → t^-1 wj}];
        ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
        TBchecks = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};

In[•]:= Short[R3l = TBchecks // TB1,2 // TB1,3 // TB2,3, 10]

Out[•]//Short= {f0 v1 - f0 u1^2 w2 - f1 u1^2 w2 + t f1 u1^2 w2 + f2 u1^2 w2 - t f2 u1^2 w2 + f0 u1 u2 w2 - f0 u1^2 w3 - f1 u1^2 w3 +
                t f1 u1^2 w3 + 2 f2 u1^2 w3 - f2 u1^2 w3 / t - t f2 u1^2 w3 - f3 u1^2 w3 + f3 u1^2 w3 / t - f2 u1 u2 w3 + t f2 u1 u2 w3 +
                f3 u1 u2 w3 - t f3 u1 u2 w3 + f0 u1 u3 w3 + f1 v1 η[1] + f2 v1 η[2] + f3 v1 η[3], <<9>>, w3 / t^2}

In[•]:= R3r = TBchecks // TB2,3 // TB1,3 // TB1,2; R3l - R3r

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Short[OCl = TBchecks // TB1,2 // TB1,3]

Out[•]//Short= {<<20>> + f2 v1 η[2] + f3 v1 η[3], <<1>>, <<48>> + <<1>>, <<5>>, <<1>>, w2 / t, w3 / t}

In[•]:= OCr = TBchecks // TB1,3 // TB1,2; OCl - OCr

Out[•]= {0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3,
         -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2, 0, 0, 0, 0, 0, 0, 0, 0}
```

The Turbo@1 Burau (Non) Representation

```
In[•]:= TB1_{i_,j_}[ℰ_] :=
          Expand[ℰ /. {
            v_k_ :> (v_k /. v_j -> (1 - t) v_i + t v_j + δ_{1,i} (v1_j - v1_1)) + δ_{k,i} (u_j - u_i) u_i w_j,
            v1_k_ :>
              (v1_k /. v1_j -> (1 - t) v1_i + t v1_j) + (t - 1) (δ_{1,i} - δ_{1,j}) (u_k /. u_j -> (1 - t) u_i + t u_j) u_i w_j,
            u_j -> (1 - t) u_i + t u_j,
            w_i -> w_i + (1 - t^-1) w_j, w_j -> t^-1 w_j}];
        TB1checks = {v_1, v_2, v_3, v1_1, v1_2, v1_3, u_1^2 w_1, u_1^2 w_2, u_1, u_2, u_3, w_1, w_2, w_3};

In[•]:= Short[R3l = TB1checks // TB1_{1,2} // TB1_{1,3} // TB1_{2,3}, 10]

Out[•]//Short= {v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,
 v_1 - t v_1 + t v_2 - v1_1 + v1_2 - u_1 u_2 w_3 + t u_1 u_2 w_3 - t u_2^2 w_3 + u_1 u_3 w_3 - t u_1 u_3 w_3 + t u_2 u_3 w_3,
 v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3 - v1_1 + v1_2 - t v1_2 + t v1_3, v1_1 - u_1^2 w_2 + t u_1^2 w_2 - u_1^2 w_3 + t u_1^2 w_3,
 v1_1 - t v1_1 + t v1_2 - u_1^2 w_2 + 2 t u_1^2 w_2 - t^2 u_1^2 w_2 - t u_1 u_2 w_2 + t^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t u_1^2 w_3 - t^2 u_1^2 w_3 -
  t u_1 u_2 w_3 + t^2 u_1 u_2 w_3, v1_1 - t v1_1 + <<29>> + t^3 u_1 u_3 w_3, u_1^2 w_1 + <<6>>, <<1>> / t - <<1>> / <<1>> + <<1>> / t,
 u_1, u_1 - t u_1 + t u_2, u_1 - t u_1 + t u_2 - t^2 u_2 + t^2 u_3, w_1 + w_2 - w_2 / t + w_3 - w_3 / t, w_2 / t - w_3 / t^2 + w_3 / t, w_3 / t^2}

In[•]:= R3r = TB1checks // TB1_{2,3} // TB1_{1,3} // TB1_{1,2}; R3l - R3r

Out[•]= {0, 0, -t u_1 u_2 w_2 + t^2 u_1 u_2 w_2 + t u_1 u_3 w_2 - t^2 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```