Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2019-10/.

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

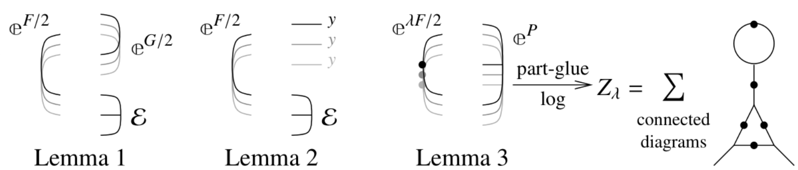
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Goals:

Implement the containers $|F,\mathcal{E}|_B := [F:\mathcal{E}]_B$ and $\langle F,\mathcal{E}\rangle_B := \langle F:\mathcal{E}\rangle_B$, their evaluator Ev_k as power series in $\hbar$ to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Generic Polynomials:

In[ ]:=
```
GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
    gc_(Sequence@@#) Times @@ (vars^#) &,
    Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
  ]];
GenericPolynomial[specs_List, vars_List, gc_] :=
 Sum[GenericPolynomial[specs[[1]], vars, gc], specs]
```

Preliminary Definitions

```
In[]:= Unprotect[Expand];
       Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
       Protect[Expand];
```

```
In[]:= CF[⟨F_, ℰ_⟩B_] := ⟨Simplify@F, Simplify@ℰ⟩B;
```

Variables and their duals:

```
In[]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
       {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i; (B_List)* := #* & /@B;
```

Act and Contract:

```
In[]:= Evk_@|F_, ℰ_|B_ := Expand[Total[
           CoefficientRules[Normal@Series[e^(B*.F.B*/2), {ħ, 0, k}], B*] /.
            (ps_ → c_) ⧴ c D[ℰ, Sequence @@ Thread[{B, ps}]]
         ] + O[ħ]^(k+1)];
       Evk_@⟨F_, ℰ_⟩B_ := Evk@|F, ℰ|B /. Alternatives @@ B → 0
```

In[]:= $\left\{\mathrm{Ev}_2@\left|\hbar\begin{pmatrix}0&1\\1&0\end{pmatrix}, e^{x y}\right|_{\{x,y\}}, \mathrm{Ev}_3@\left\langle\hbar\begin{pmatrix}0&1\\1&0\end{pmatrix}, e^{3 x y}\right\rangle_{\{x,y\}}\right\}$

Out[]= $\left\{e^{x y}+\left(e^{x y}+e^{x y} x y\right) \hbar+\left(e^{x y}+2 e^{x y} x y+\frac{1}{2} e^{x y} x^2 y^2\right) \hbar^2+O[\hbar]^3, 1+3 \hbar+9 \hbar^2+27 \hbar^3+O[\hbar]^4\right\}$

Implementing / Testing Lemma 1

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$$

In[∘]:=
```
{p = 10, B = {x}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ {{f}}, G = {{g}}, ℰ = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Ev_p@⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{10, \{x\}, \{\{1\}\}, \{\{f\,\hbar\}\}, \{\{g\}\}, c_0 + x\,c_1 + x^2 c_2 + x^3 c_3 + x^4 c_4\}$

Out[∘]= $$c_0 + \left(\frac{1}{2} f g c_0 + f c_2\right)\hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g c_2 + 3 f^2 c_4\right)\hbar^2 + \left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g c_4\right)\hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4\right)\hbar^4 + \left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4\right)\hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4\right)\hbar^6 + \left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4\right)\hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024}\right)\hbar^8 + \left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048}\right)\hbar^9 + \left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768}\right)\hbar^{10} + O[\hbar]^{11}$$

Out[∘]= True

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12
         f12 f22), G = (g11 g12
                        g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, 𝓔 ⅇ^(B.G.B/2)⟩_B]
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) 𝓔⟩_B;
HL[lhs == rhs]
```

Out[∘]=
$$\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$$

Out[∘]=
$$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right) \hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= True

In[∘]:=
```
DeGauss@⟨F_, 𝓔_⟩_B_ := Module[{𝓘, Q, G, M, Δ},
   𝓘 = Echo@IdentityMatrix@Length@B;
   Q = Echo@Log[Normal[𝓔] /. ϵ → 0];
   G = Echo@Table[∂_{i,j}Q, {i, B}, {j, B}];
   M = Echo@Inverse[𝓘 - G.F];
   Δ = Echo@Simplify@Det@M;
   CF@⟨F.M, Δ^(1/2) 𝓔 ⅇ^(-B.G.B/2)⟩_B
  ]
```

In[∘]:=
```
{p = 2, B = {x, y}, F = ℏ (f11 f12
                           f12 f22),
   G = (g11 g12
        g12 g22),  𝓔 = 1 + ϵ GenericPolynomial[{d, 0, 2}, B, c]};
lhs = ⟨F, 𝓔 ⅇ^(B.G.B/2)⟩_B
rhs = DeGauss@⟨F, 𝓔 ⅇ^(B.G.B/2)⟩_B
HL[Ev_p@lhs == Ev_p@rhs]
```

Out[∘]=
$$\left\langle \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},\ e^{\frac{1}{2}(x (x g_{11}+y g_{12})+y (x g_{12}+y g_{22}))} \left(1 + \epsilon \left(c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\right)\right) \right\rangle_{\{x,y\}}$$

» $\{\{1, 0\}, \{0, 1\}\}$

» $\mathrm{Log}\left[e^{\frac{1}{2}(x(x g_{11}+y g_{12})+y(x g_{12}+y g_{22}))}\right]$

» $\{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}$

» $$\left\{\left\{\frac{1-\hbar f_{12} g_{12}-\hbar f_{22} g_{22}}{1-\hbar f_{11} g_{11}-2\hbar f_{12} g_{12}+\hbar^2 f_{12}^2 g_{12}^2-\hbar^2 f_{11} f_{22} g_{12}^2-\hbar f_{22} g_{22}-\hbar^2 f_{12}^2 g_{11} g_{22}+\hbar^2 f_{11} f_{22} g_{11} g_{22}},\right.\right.$$
$$\left.\frac{\hbar f_{12} g_{11}+\hbar f_{22} g_{12}}{1-\hbar f_{11} g_{11}-2\hbar f_{12} g_{12}+\hbar^2 f_{12}^2 g_{12}^2-\hbar^2 f_{11} f_{22} g_{12}^2-\hbar f_{22} g_{22}-\hbar^2 f_{12}^2 g_{11} g_{22}+\hbar^2 f_{11} f_{22} g_{11} g_{22}}\right\},$$
$$\left\{\frac{\hbar f_{11} g_{12}+\hbar f_{12} g_{22}}{1-\hbar f_{11} g_{11}-2\hbar f_{12} g_{12}+\hbar^2 f_{12}^2 g_{12}^2-\hbar^2 f_{11} f_{22} g_{12}^2-\hbar f_{22} g_{22}-\hbar^2 f_{12}^2 g_{11} g_{22}+\hbar^2 f_{11} f_{22} g_{11} g_{22}},\right.$$
$$\left.\left.\frac{1-\hbar f_{11} g_{11}-\hbar f_{12} g_{12}}{1-\hbar f_{11} g_{11}-2\hbar f_{12} g_{12}+\hbar^2 f_{12}^2 g_{12}^2-\hbar^2 f_{11} f_{22} g_{12}^2-\hbar f_{22} g_{22}-\hbar^2 f_{12}^2 g_{11} g_{22}+\hbar^2 f_{11} f_{22} g_{11} g_{22}}\right\}\right\}$$

» $$\frac{1}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))}$$

Out[∘]=
$$\Bigg\langle\left\{\left\{\frac{\hbar(\hbar f_{12}^2 g_{22}+f_{11}(1-\hbar f_{22} g_{22}))}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))},\right.\right.$$
$$\left.\frac{\hbar(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12})}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))}\right\},$$
$$\left\{\frac{\hbar(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12})}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))},\right.$$
$$\left.\left.\frac{\hbar(\hbar f_{12}^2 g_{11}+f_{22}(1-\hbar f_{11} g_{11}))}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))}\right\}\right\},$$
$$\sqrt{\frac{1}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2 (g_{12}^2-g_{11} g_{22})+\hbar f_{11}(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22}))}}$$
$$\left(1+\epsilon\, c_{0,0}+y\,\epsilon\, c_{0,1}+y^2\,\epsilon\, c_{0,2}+x\,\epsilon\, c_{1,0}+x\,y\,\epsilon\, c_{1,1}+x^2\,\epsilon\, c_{2,0}\right)\Bigg\rangle_{\{x,y\}}$$

Out[∘]= True

Testing Lemma 2

Lemma 2. $\left\langle F:\mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+F y_B}\right\rangle_B.$

In[∘]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = GenericPolynomial[{d, 0, 3}, B, c]}
```

Out[∘]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$c_{0,0}+z_2 c_{0,1}+z_2^2 c_{0,2}+z_2^3 c_{0,3}+z_1 c_{1,0}+z_1 z_2 c_{1,1}+z_1 z_2^2 c_{1,2}+z_1^2 c_{2,0}+z_1^2 z_2 c_{2,1}+z_1^3 c_{3,0}\}$

In[∘]:= $\texttt{Timing}\left[\texttt{lhs} = \texttt{Ev}_{\texttt{p}}\texttt{@}\left\langle \texttt{F}, \mathcal{E}\, e^{\texttt{Y}\cdot\texttt{B}}\right\rangle_{\texttt{B}}\right]$

Out[∘]= $\{0.515625,$

$$c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + 2 f_{12}^2 y_1 y_2 c_{1,1} + f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + 3 f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + 3 f_{11} f_{12} y_1 c_{2,1} + 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$$

In[∘]:= $\texttt{Timing}\left[\texttt{rhs} = \texttt{Expand}\left[\texttt{Series}\left[e^{\texttt{Y}\cdot\texttt{F}\cdot\texttt{Y}/2}\,\texttt{Ev}_{\texttt{p}}\texttt{@}\langle \texttt{F}, \mathcal{E}\ \texttt{/. Thread}[\texttt{B} \to \texttt{B} + \texttt{F.Y}]\rangle_{\texttt{B}}, \{\hbar, 0, \texttt{p}\}\right]\right]\right]$

Out[∘]= $\{0.0625,$

$$c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + 2 f_{12}^2 y_1 y_2 c_{1,1} + f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + 3 f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + 3 f_{11} f_{12} y_1 c_{2,1} + 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$$

```
In[]:= Timing[rhs = Ev_p@⟨F, e^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]⟩_B]
```

Out[]= $\{0.125,\ c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + 2 f_{12}^2 y_1 y_2 c_{1,1} + f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + 3 f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + 3 f_{11} f_{12} y_1 c_{2,1} + 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$

```
In[]:= HL[lhs == rhs]
```

Out[]= True

Testing Lemma 3

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

```
In[]:= {n = 2, p = 2, B = Table[b_i, {i, n}],
         F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[]= $\{2,\ 2,\ \{b_1, b_2\},\ \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},\ c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\}$

In[ ]:= **Z = PowerExpand@Expand@Log[Ev_p@ |λ F, e^P|_B]**

Out[ ]=

$$\left(c_{0,0}+b_2 c_{0,1}+b_2^2 c_{0,2}+b_1 c_{1,0}+b_1 b_2 c_{1,1}+b_1^2 c_{2,0}\right)+$$
$$\Big(\frac{1}{2}\lambda f_{22} c_{0,1}^2+\lambda f_{22} c_{0,2}+2\lambda b_2 f_{22} c_{0,1} c_{0,2}+2\lambda b_2^2 f_{22} c_{0,2}^2+\lambda f_{12} c_{0,1} c_{1,0}+2\lambda b_2 f_{12} c_{0,2} c_{1,0}+$$
$$\frac{1}{2}\lambda f_{11} c_{1,0}^2+\lambda f_{12} c_{1,1}+\lambda b_2 f_{12} c_{0,1} c_{1,1}+\lambda b_1 f_{22} c_{0,1} c_{1,1}+2\lambda b_2^2 f_{12} c_{0,2} c_{1,1}+$$
$$2\lambda b_1 b_2 f_{22} c_{0,2} c_{1,1}+\lambda b_2 f_{11} c_{1,0} c_{1,1}+\lambda b_1 f_{12} c_{1,0} c_{1,1}+\frac{1}{2}\lambda b_2^2 f_{11} c_{1,1}^2+$$
$$\lambda b_1 b_2 f_{12} c_{1,1}^2+\frac{1}{2}\lambda b_1^2 f_{22} c_{1,1}^2+\lambda f_{11} c_{2,0}+2\lambda b_1 f_{12} c_{0,1} c_{2,0}+4\lambda b_1 b_2 f_{12} c_{0,2} c_{2,0}+$$
$$2\lambda b_1 f_{11} c_{1,0} c_{2,0}+2\lambda b_1 b_2 f_{11} c_{1,1} c_{2,0}+2\lambda b_1^2 f_{12} c_{1,1} c_{2,0}+2\lambda b_1^2 f_{11} c_{2,0}^2\Big)\hbar+$$
$$\Big(\lambda^2 f_{22}^2 c_{0,1}^2 c_{0,2}+\lambda^2 f_{22}^2 c_{0,2}^2+4\lambda^2 b_2 f_{22}^2 c_{0,1} c_{0,2}^2+4\lambda^2 b_2^2 f_{22}^2 c_{0,2}^3+2\lambda^2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0}+$$
$$4\lambda^2 b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0}+\lambda^2 f_{12}^2 c_{0,2} c_{1,0}^2+\lambda^2 f_{12} f_{22} c_{0,1}^2 c_{1,1}+2\lambda^2 f_{12} f_{22} c_{0,2} c_{1,1}+$$
$$6\lambda^2 b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1}+2\lambda^2 b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1}+8\lambda^2 b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1}+4\lambda^2 b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1}+$$
$$\lambda^2 f_{12}^2 c_{0,1} c_{1,0} c_{1,1}+\lambda^2 f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1}+4\lambda^2 b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1}+2\lambda^2 b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1}+$$
$$2\lambda^2 b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1}+\lambda^2 f_{11} f_{12} c_{1,0}^2 c_{1,1}+\frac{1}{2}\lambda^2 f_{12}^2 c_{1,1}^2+\frac{1}{2}\lambda^2 f_{11} f_{22} c_{1,1}^2+\lambda^2 b_2 f_{12}^2 c_{0,1} c_{1,1}^2+$$
$$\lambda^2 b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2+2\lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2+3\lambda^2 b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2+2\lambda^2 b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2+$$
$$6\lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2+\lambda^2 b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2+2\lambda^2 b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2+\lambda^2 b_1 f_{12}^2 c_{1,0} c_{1,1}^2+$$
$$\lambda^2 b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2+\lambda^2 b_2^2 f_{11} f_{12} c_{1,1}^3+\lambda^2 b_1 b_2 f_{12}^2 c_{1,1}^3+\lambda^2 b_1 b_2 f_{11} f_{22} c_{1,1}^3+\lambda^2 b_1^2 f_{12} f_{22} c_{1,1}^3+$$
$$\lambda^2 f_{12}^2 c_{0,1}^2 c_{2,0}+2\lambda^2 f_{12}^2 c_{0,2} c_{2,0}+4\lambda^2 b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0}+4\lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0}+$$
$$4\lambda^2 b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0}+8\lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0}+2\lambda^2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0}+$$
$$4\lambda^2 b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0}+4\lambda^2 b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0}+\lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0}+2\lambda^2 f_{11} f_{12} c_{1,1} c_{2,0}+$$
$$2\lambda^2 b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0}+4\lambda^2 b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0}+2\lambda^2 b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0}+$$
$$4\lambda^2 b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0}+12\lambda^2 b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0}+4\lambda^2 b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0}+$$
$$4\lambda^2 b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0}+2\lambda^2 b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0}+6\lambda^2 b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0}+$$
$$\lambda^2 b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0}+6\lambda^2 b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0}+3\lambda^2 b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0}+2\lambda^2 b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0}+$$
$$\lambda^2 f_{11}^2 c_{2,0}^2+4\lambda^2 b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2+8\lambda^2 b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2+4\lambda^2 b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2+$$
$$4\lambda^2 b_1 f_{11}^2 c_{1,0} c_{2,0}^2+4\lambda^2 b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2+8\lambda^2 b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2+4\lambda^2 b_1^2 f_{11}^2 c_{2,0}^3\Big)\hbar^2+O[\hbar]^3$$

In[ ]:= **Z /. λ → 0**

Out[ ]=

$$\left(c_{0,0}+b_2 c_{0,1}+b_2^2 c_{0,2}+b_1 c_{1,0}+b_1 b_2 c_{1,1}+b_1^2 c_{2,0}\right)+O[\hbar]^3$$

In[ ]:= **(Z /. λ → 0) - P**

Out[ ]=

$$O[\hbar]^3$$

In[∘]:= **lhs = ∂$_\lambda$ Z**

Out[∘]=

$$\Big(\tfrac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \tfrac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} + b_1 f_{12} c_{1,0} c_{1,1} + \tfrac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \tfrac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} + 4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big) \hbar +$$

$$\Big(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + 8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} + 12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 + \lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 + 2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1}^2 c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} + 8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + 2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + 2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + 8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3\Big) \hbar^2 + O[\hbar]^3$$

In[∘]:= **Short[rhs = Expand@Sum[($\partial_{b1,b2}$(B.F.B)) ($\partial_{b1,b2}$Z + (∂_{b1}Z) (∂_{b2}Z)) / 4, {b1, B}, {b2, B}]]**

Out[∘]//Short=

$$\Big(\tfrac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big) \hbar + \big(\ll 1 \gg\big) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4$$

In[∘]:= **HL[Normal[lhs - rhs] == 0]**

Out[∘]= True

In[∘]:= **Z /. λ → 1 /. Alternatives @@ B → 0**

Out[∘]=

$$c_{0,0} + \Big(\tfrac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \tfrac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0}\Big) \hbar +$$

$$\Big(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \tfrac{1}{2} f_{12}^2 c_{1,1}^2 + \tfrac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2\Big) \hbar^2 + O[\hbar]^3$$