Knot

(Rec-)Tangle invariants and Gaussians

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Abstract

Motivated by the algebra of universal knot invariants and quasi-triangular Hopf algebras we give new method for setting up and computing with knots and tangles and natural operations on them. Our main example is constructed using a version of the Drinfeld double construction and is likely equivalent to the universal quantum sl_2 invariant. We show that approximations to it are still powerful knot invariants that can be computed in polynomial time.

1 Introduction

A little OU drama introducting algebras and the Drinfeld double? Introduce PBW basis and Gaussians. Some concrete computations using simplified Gaussians? State some main results and conjectures in an accessible way.

Plan of the rest of the paper, Answer the following questions:

1. Invariant of what? Rec-tangles
2. With values where? two-step Gaussians
3. How to construct? Drinfeld double construction for quasi-triangular categories.
4. More details further directions? Give remaining proofs, odds and ends.

2 (rec-)Tangle invariant

2.1 Rec-tangles

A common approach to finding invariants of knots is to use Reidemeister's theorem to find functions on knot diagrams that are invariant under the Reidemeister moves. The hardest part is to deal with the third Reidemeister move which comes down to solving the Yang-Baxter equation. In the context of Hopf algebras this equation factors in terms of simpler equations that say that the Hopf algebra is quasi-triangular. In this section we outline a theory of generalized knot diagrams called *rec-tangles*, that may be viewed as a topological counterpart to the theory of quasi-triangular Hopf algebras.

An important feature of rec-tangles is that they allow us to discuss both knots, tangles and the common operations on them on the same footing. Much like tensors in algebra allow one to view vectors and a linear maps as instances of the same thing. For example, the most common operations on strands of knots and tangles are merging, doubling, creation, deletion and reversal. The rec-tangles corresponding to these operations are shown in Figure 2. In fact their Hopf algebraic counterparts are multiplication, co-multiplication, unit, co-unit and antipode respectively.

We generalize this to a theory of immersed intervals in a surface endowed with a vector field. Instead of keeping track of the sign of each crossing in the diagram we keep track of the (relative) rotation number of the tangent vector with respect to the reference vector field. This needs to be done anyway so why not put all information into the rotation numbers.

Definition 1. *(relative rotation number)*
Consider two vector fields w, v defined along an oriented curve in an oriented surface, such that v, w are independent at the endpoints. Define a function $\varphi_{w,v}$ on the curve with values in the four point set[1] $\{\pm 1, \pm \mathrm{i}\}$ *by*

$$\varphi_{w,v}(p) = \begin{cases} \pm 1 & \textit{if } w(p) = \pm\lambda v(p), \lambda > 0 \\ \pm \mathrm{i} & \textit{if } (w(p), v(p)) \textit{is a } \pm \textit{ basis} \end{cases}$$

The (relative) rotation number of w with respect to v along the curve is the sum of all occurrences of triples of adjacent values of φ where the triples $\pm(-\sigma\mathrm{i}, -1, \sigma\mathrm{i})$ count as $\frac{\sigma}{2}$ for $\sigma \in \{-1, 1\}$. The rotation number of oriented curve γ with respect to v is the rotation number of the tangent vector field $\dot{\gamma}$ with respect to v.

[1]this set with the topology that makes φ continous is known as the pseudo-circle, weak homotopy equivalent to the circle.

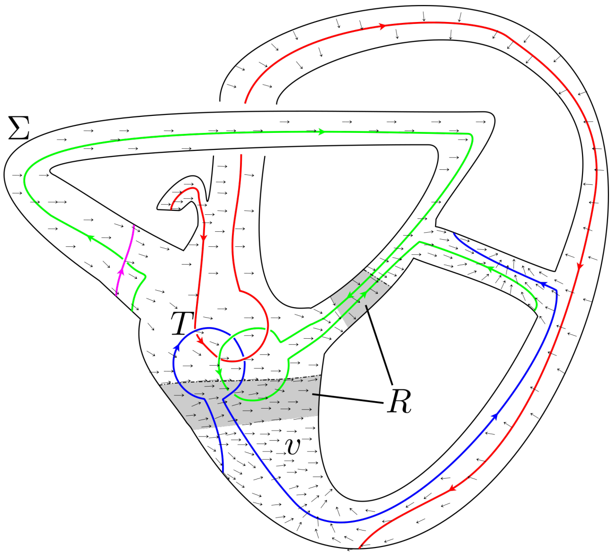

Figure 1: A rec-tangle with four strands (purple, red, green, blue) and two rectangles.

Definition 2. *(Rec-tangles)*
A rec-tangle is a quadruple (Σ, v, R, T) *consisting of a surface* Σ*, a vector field* v *on* Σ*, a finite set* R *of distinguished bands (rectangles) and the oriented image* T *of a generic relative immersion of finitely many copies of* $([0,1], \partial[0,1])$ *into* $(\Sigma, \partial\Sigma)$*. The immersed intervals are known as the strands of* T*.*

1. *The surface* Σ *is compact, oriented with boundary.*
2. *A rectangle is an embedded copy of* $[0,1]^2$ *in* Σ *if two opposite edges are embedded in* $\partial\Sigma$*. One of the remaining two edges is called the head the other the foot.*
3. *Strands of* T *may only intersect the boundary of a rectangle transversely at the head and foot.*
4. *The vector field* v *is required to have the following properties:*
   - *(a)* v *is continuous and nowhere vanishing.*
   - *(b) Neither of the two tangent vectors at a singularity of* T *are parallel to* $\pm v$*.*
   - *(c) Also,* v *is tangent to the head and foot arcs of the rectangles in* R *and if* o *points out of the rectangle the pair* (v, o) *is a positive basis of the tangent space for the head and a negative basis for the foot.*
   - *(d) Around each endpoint of a strand of* T *there is an interval in* $\partial\Sigma$ *where* v *is tangent to* $\partial\Sigma$ *and if* t *is the tangent vector to the strand,* (v, t) *is a positive basis at the endpoints.*

In drawing pictures the vector field v is usually assumed to be constant and equal to the first standard basis vector of $\mathbb{R}^2$. The orientation of Σ is usually the standard orientation of $\mathbb{R}^2$. The rectangles are drawn in grey and the head is marked with a dashed line. To suggest the connection to tangle diagrams explained below the singularities of T are often drawn as 'crossings' with the convention that the over-strand always goes lower left to upper right or vice versa. Some of the basic rec-tangles are depicted in figure 2.

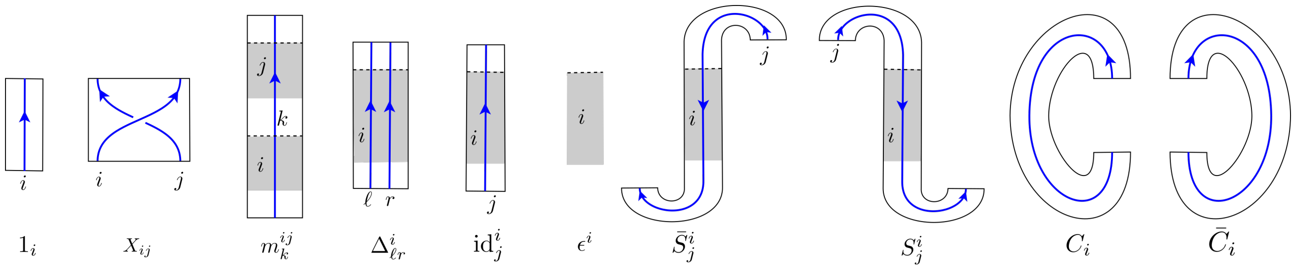

Figure 2: Basic rec-tangles. The vector field v is constantly in the direction of the positive x-axis.

The above definition is to be taken up to orientation preserving diffeomorphism of the quadruple (Σ, v, R, T) where v is transformed by the derivative. Next we define two unary operations on rec-tangles that should resemble surgery on a cylinder in a 3-manifold with a tangle inside. Roughly speaking we take a rectangle and identify it with a tubular neighborhood of a strand disjoint form it. Technically it is easier to make a parallel of the strand and then append or prepend the contents of the rectangle as shown in Figure 4. A concrete example is shown in Figure 5.

Definition 3. *(pre- and post-contraction of rec-tangles)*
Suppose we have a rectangle r disjoint from strand $s \subset T$ in rec-tangle (Σ, v, R, T). The pre/post-contraction *of r and s is the rec-tangle $\Sigma' = (\Sigma', v', R', T')$ defined as follows.*

- *The surface Σ' is obtained from Σ by choosing embedded closed intervals I_{head} and I_{foot} in $\partial\Sigma$ containing the end points of s where v is tangent to $\partial\Sigma$. Now cut Σ along the head/foot of r and identify r along its head/foot with I_{foot}/I_{head} so that v remains well-defined. Similarly, the other side of the cut should be identified with I_{head}/I_{foot}. Corners resulting from the gluing should be smoothed.*
- *Next set $R' = R - r$.*
- *The vector field v' is equal to v.*
- *Finally to get T' we split s into as many parallel strands as intersect the head/foot of r, making sure the new end points are in $I_{foot} \cup I_{head}$. Near the singularities of s the parallel strands should look like one of the pictures in Figure 3 and corresponding parallel arcs between singularities and or rectangles should have equal rotation numbers. Near the head/foot we should adjust the new parallel strands to merge smoothly with the strands from r entering the head/foot curve the strands should be oriented to match those from r.*

too wide

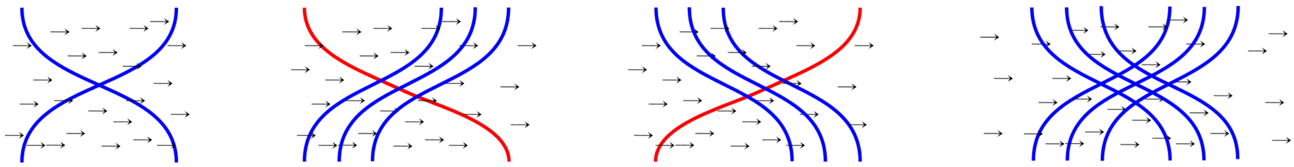

Figure 3: Local models for the n-parallel (shown in blue) of a strand near a singularity of the original strand. Another strand is drawn in red. Here we illustrated the case $n = 3$.

Important special cases of contraction arise when one considers the union of a rec-tangle Σ with strands s, t with the basic rec-tangle m_u^{st}. The effect of contraction strand s with rectangle s and strand t with rectangle t is simply connecting the end of strand s to the start of strand t with a band, joining the two strands and calling the result u. Likewise, after contracting the union of Σ and Δ_{ab}^r along strand r and rectangle r is a copy of Σ where the strand r is now doubled and the two new strands are called a, b. Taking union and contracting with the other basic tangles gives similarly simple results that should remind one of the Hopf algebra operations.

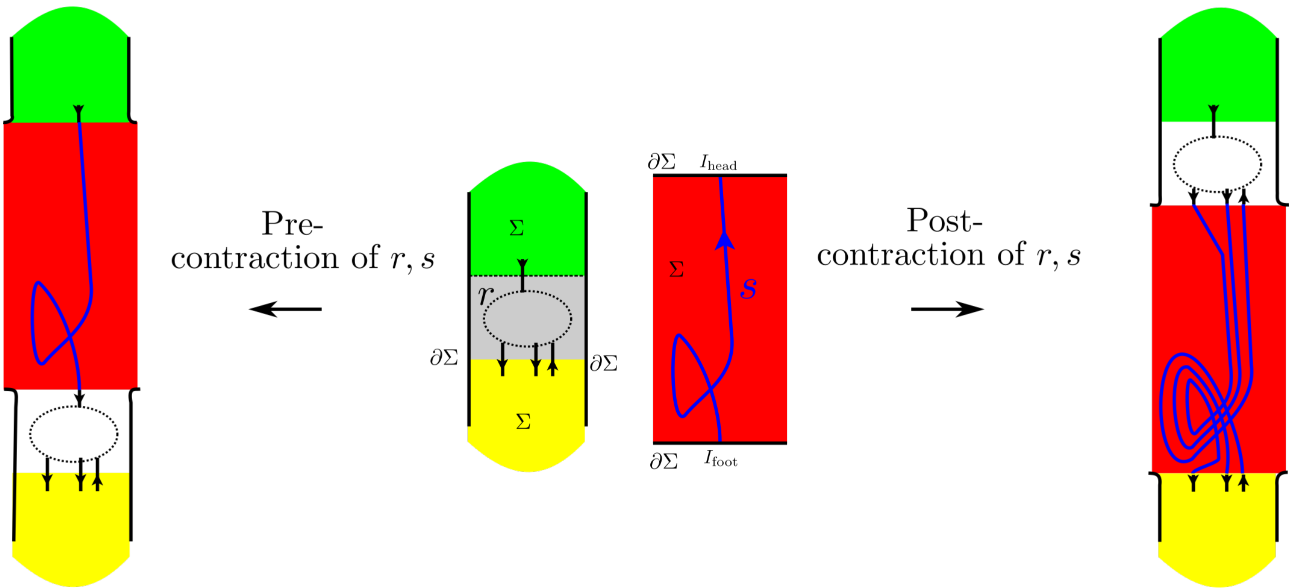

Figure 4: Pre- and post-Contraction along rectangle r (grey) and strand s (blue). To keep track of the pieces of the surface Σ we have colored them green, yellow and red.

Definition 4. *(Equivalence of rec-tangles)*
Consider an equivalence relation $\sim$ on rec-tangles generated by the following equivalences:

1. *Pre- and post contraction should yield equivalent rec-tangles.*
2. *Contracting two equivalent rec-tangles along corresponding indices yields equivalent results.*

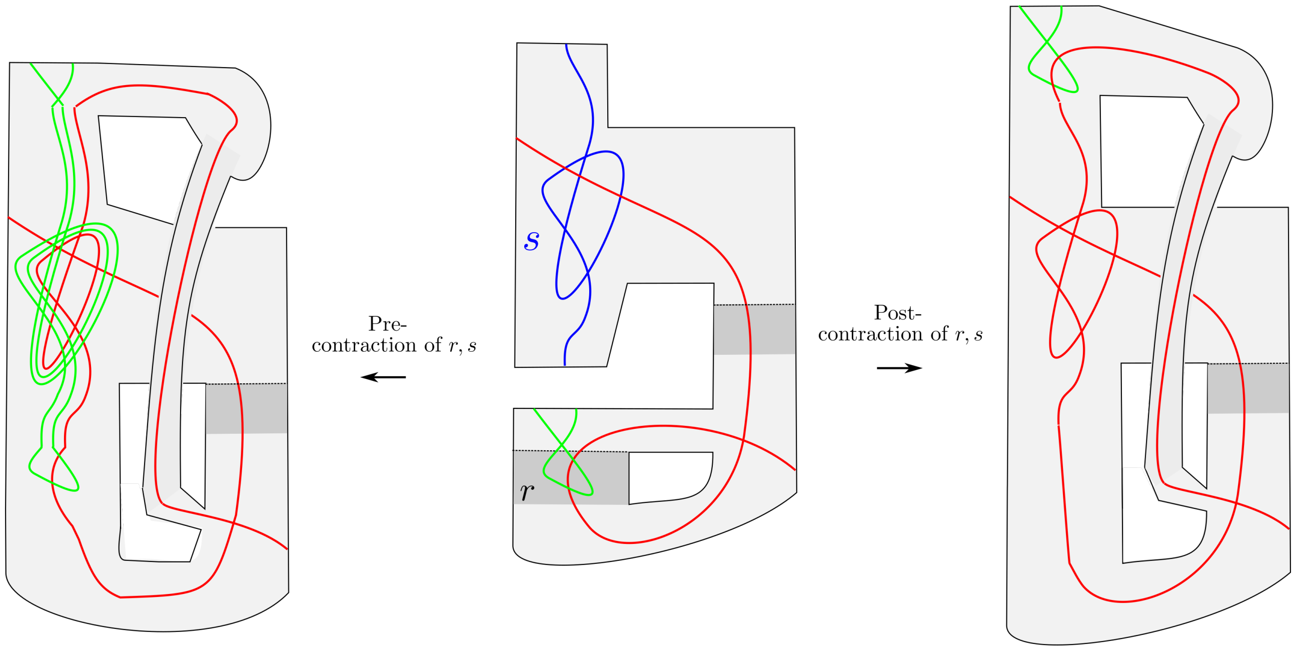

Figure 5: Pre- and post-contraction along r, s on the example rec-tangle shown in the middle.

3. *Contraction with* id_i *yields an equivalent rec-tangle.*
4. *Attaching/removing a* 0 *or* 1*-handle (disk or band) to the* $\partial\Sigma$ *away from* T *and the rectangles as long as it is possible to extend vector field* v *in a differentiable way.*
5. *Homotopy of the vector field* v *preserving the required properties of* v *in Definition 2 at all stages and preserving the rotation numbers along all arcs of* T *in the complement of singularities and rectangles.*

For convenience we set up a category of equivalence classes of labeled rec-tangles as follows.

Definition 5. *The category* $\mathcal{R}$ *of rec-tangles has finite sets for objects.* $Hom_{\mathcal{R}}(R, C) = \mathcal{R}(R, C)$ *is the set of equivalence classes of rec-tangles whose rectangles are labelled by* R *and whose strands are labeled by* C*. Whenever* $R \cap R' = \emptyset = C \cap C'$ *(disjoint) union defines map* $\cup : \mathcal{R}(R, C) \times \mathcal{R}(R', C')$*. Also composition is defined to be the contraction of* $f \cup g$ *along all pairs of labels that coincide. We will often use the notation* $g /\!\!/ f = f \circ g$ *for compositions and union will usually be abbreviated as a juxtaposition.*

With the above conventions and setting $m_k^{a,b,c,d,\dots} = m_k^{a,b} /\!\!/ m_k^{k,c} /\!\!/ m_k^{k,d} /\!\!/ \dots$, rec-tangles may be written concisely and precisely as a composition of basic rec-tangles. For example, a long trefoil knot $\mathcal{T}$ is written below, see also Figure 6

$$\mathcal{T}_1 = X_{51} X_{26} X_{73} C_4 /\!\!/ m_1^{12345678}$$

Another rec-tangle that will be used when discussing Seifert surfaces and Alexander polynomials is $\mathcal{B}_k^{ij}$ shown on the left of the same figure. We can similarly express $\mathcal{B}$ as

$$\mathcal{B}_k^{ij} = \Delta^i_{\ell_1 r_1} \Delta^j_{\ell_2 r_2} /\!\!/ S^{r_1}_{r_1} \bar{S}^{r_2}_{r_2} C_1 C_2 /\!\!/ m_k^{\ell_1, r_2, 1, 2, r_1, \ell_2}$$

The importance of rec-tangles to knot theory comes from the fact that (isotopy equivalence classes of) ordinary tangles without closed components inject into rec-tangles. Recall that by an ordinary tangle we mean properly embedded intervals and circles into a 3-ball with endpoints on a distinguished simple closed curve on the boundary. Cutting a knot to turn it into a single component tangle means knots also inject into rec-tangles.

As suggested by the drawings, the singularities of T may be interpreted as crossings of a tangle diagram with the convention that the over-strand always goes from bottom left to top right or vice versa. This convention may seem strange but is economical since we need to keep track of rotation numbers along the edges of a tangle diagram anyway.

Lemma 1. *Any oriented ordinary tangle without closed components may be presented by a blackboard* 0*-framed tangle diagram in a disk such that the rotation number with respect to the constant vector field* e_1 *is* 0 *along each component. This provides an injective map of isotopy classes of oriented tangles into rec-tangles (without rectangles).*

Proof. The requirement on the rotation numbers is no real restriction since the sum of the rotation number and the framing is always even in the plane. Consequently one can find a representant whose framing is zero and make the rotation number zero as well by adding double curls that have rotation number ± 2 and have

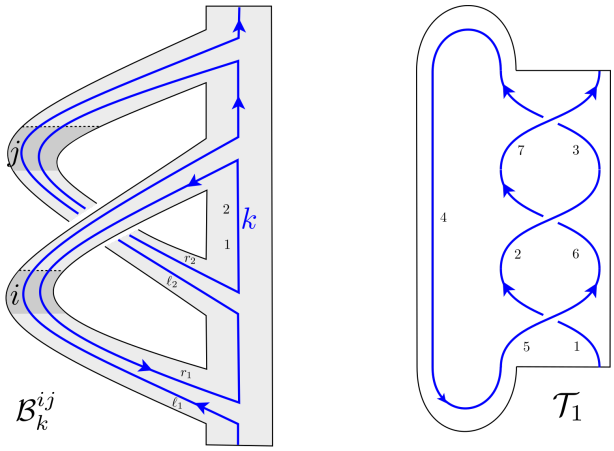

Figure 6: (left) The rec-tangle $\mathcal{B}$ (right) A long trefoil knot $\mathcal{T}$.

framing 0. Actually to construct plane tangle diagrams one only needs the crossings $X^{\pm}, S^{\pm 2}$ and merging m_k^{ij}.

Following Ohtsuki we may present any tangle using a so called oriented sliced tangle diagram. Any such diagram can be turned into a rec-tangle in our sense by making the vector field v point constantly in the positive x-direction, the end points of tangles should point upwards and the over-pass should go bottom left to top right or vice versa.

If two such diagrams describe the same tangle then they must be related by the sliced Reidemeister moves shown on p.47 of Ohtsuki. Translated into our notation isotopy of tangles comes down to proving the equivalences of rec-tangles shown in Figure 7. All these equivalences except the bottom left version of

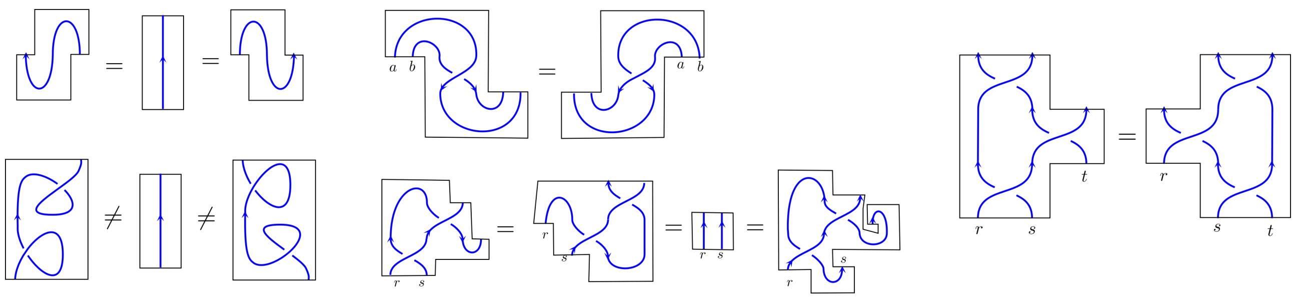

Figure 7: Ohtsuki's sliced Reidemeister moves in the language of rec-tangles. Actually the bottom left does not hold in the category of rectangles!

Reidemeister 1 preserve the condition that the rotation number is 0. Actually the bottom left relation never has to be used to describe an isotopy between two diagrams of rotation number 0. The other relations follow fairly quickly from the definition of equivalence of rec-tangles as shown below.

First, the two sides of the braid-like Reidemeister 3 may be viewed as the pre- and post-contraction of a crossing X_{ij} with a crossing X_{ab} where the latter is inside a (disjoint) rectangle labeled j as shown in figure 8. Similarly for braid-like Reidemeister 2 where now we take a u-turn inside the rectangle labelled j. The other Reidemeister 2 is obtained by composing with the antipode S on strand labelled s. Finally the remaining straightening move follows from our definition of equivalence (homotopy) of the vector field.

□

The fact that the rec-tangles shown in the bottom left of Figure 7 are not equal will be addressed later. We will eventually construct an invariant of rec-tangles that does take the same value on all three rec-tangles shown there. However quotienting out by this relation seems unnecessary since dealing with such curls is a local affair.

Even though tangles inject into rec-tangles, the composition of rec-tangles more powerful than the one commonly found in the usual tangle category. Composition in the tangle category can be implemented by composing the union of two tangles, viewed as rec-tangles with a couple of suitably labeled m_k^{ij} rec-tangles. Rec-tangles treat tangles and operations on them on an equal footing. For example $\Delta^i_{\ell r}$ doubles strand i, marking the resulting strands ℓ, r. The rec-tangle S^i_j reverses a strand.

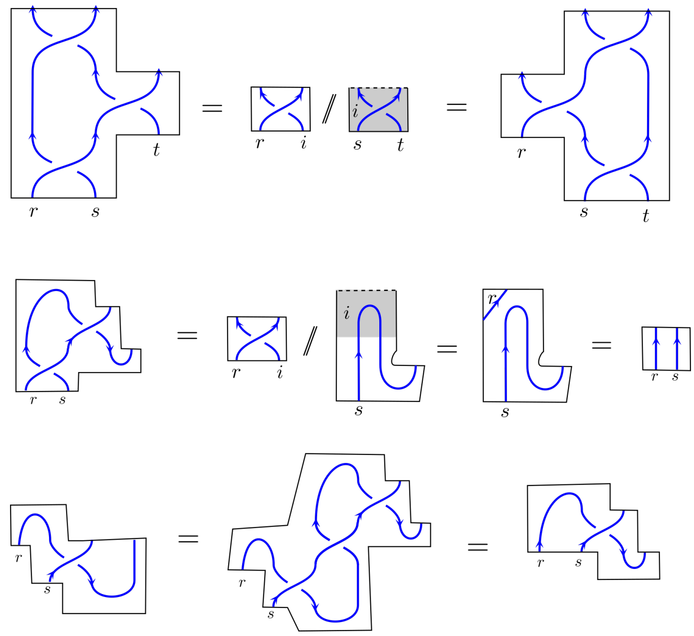

Figure 8: Reidemeister proofs.

2.2 Perturbed Gaussians

The invariant will take values in a category whose morphisms are very special power series that we call two step docile perturbed Gaussians. Before giving their definition we consider a simpler type of perturbed docile Gaussian series in general.

2.2.1 General Gaussians

We work over a field W of characteristic 0.

Definition 6. *Given finite sets I, J and variables $z = (z_j)_{j\in J}$ and $\zeta = (\zeta_i)_{i\in I}$, a (docile perturbed) Gaussian is an expression $e^{\zeta Q z}\sum_k P_k\epsilon^k$. Here $\zeta Q z = \sum_{i\in I, j\in J}\zeta_i Q_{ij} z_j$ for some coefficients $Q_{ij}\in W$. And $P_k \in W[\zeta, z]$ is of degree at most $2k$ in ζ and also at most of degree $2k$ in z.*

Note that the set of Gaussians is closed under products but not sums.

Definition 7. *For any $F \in W[\![\epsilon, z, \zeta]\!]$ the contraction in the variable ζ_k is defined (provided the result converges) as $\langle F\rangle_{\zeta_k} = (F|_{\zeta_k\mapsto\partial_{z_k}})|_{z_k\mapsto 0}$.* Nobody will understand this.

More generally we define recursively $\langle F\rangle_{v_1,\dots,v_g} = \langle\langle F\rangle_{v_1,\dots v_{g-1}}\rangle_{v_g}$. Also $\langle F\rangle_K$ means contraction along all variables ζ_i for $i\in K$.

So we replace the variable ζ_k by ∂_{z_k}, differentiate assuming the derivatives always are ordered first and then evaluate at $z_k = 0$. For example $\langle\zeta_1 z_2 + 2\zeta_2 z_2^2 + 3z_1 z_2\zeta_2\rangle_{\zeta_2} = 3z_1$. In the above the order of carrying out the contractions not important.

The crucial observation is that Gaussians are closed under contraction. This follows from

Theorem 1. *Suppose $P\in W[\![z]\!][\zeta]$ and the labels of the variables are $I = J$ in the above and $w\in W$:*

$$\langle P(z,\zeta)e^{w+\lambda\ell+\zeta l+\zeta Q z}\rangle_J = \det(\tilde{Q})e^{w+\lambda\tilde{Q}\ell}\langle P(\tilde{Q}(z+\ell), \zeta+\lambda\tilde{Q})\rangle_J$$

whenever the right hand side with $\tilde{Q} = (1-Q)^{-1}$ exists.

Proof. See section Extras. □

From the formula in the theorem it is clear that for any docile perturbed Gaussian G its contraction $\langle G\rangle_S$ is still docile perturbed Gaussian whenever it is defined. The same is true for the product of two docile Gaussians, but usually not for their sum! The reason for allowing power series in z will become clear in the next paragraph.

2.2.2 Two step Gaussians

The type of Gaussians relevant for our main result are slightly more complicated. We now have four variables $\alpha, a, \tau, t, \eta, y, \xi, x$ where the Greek variables have labels in a set I and the Latin letters have labels in set J.

Basically we consider expressions that are Gaussian in two ways simultaneously.

Definition 8. *For any finite sets I, J define the set of (docile perturbed) two step Gaussians $\mathcal{G}(I,J)$ to be subset of $\mathbb{Q}[\![a,\alpha,t,\tau,x,\xi,y,\eta]\!][\![\epsilon]\!]$ of expressions of the form $e^{(a,\tau)L\alpha t+(y,\xi)W\binom{\eta}{x}}P$. Here $\mathcal{A}=(e^{\alpha_j})_{j\in J}$, $T=(e^{-t_i})_{i\in I}$ and the matrices L,W have entries in $\mathbb{Q}$, $\mathbb{Q}(\mathcal{A},T^{\frac{1}{2}})$ respectively. Finally $P=\sum_k P_k\epsilon^k$ where $P_k\in\mathbb{Q}(\mathcal{A},T^{\frac{1}{2}})[\tau,a,x,\eta,\xi,y]$ with $2\deg_{a,\tau}P_k+\deg_{x,\eta}P_k+\deg_{y,\xi}P_k\le 4k$.*

Two step Gaussians may be contracted using the Theorem 1 twice. First we contract the variables τ, a over field $\mathbb{Q}$, considering the other variables as constants. Second the contraction is taken in the variables η, x, now over the field $\mathbb{Q}(\mathcal{A},T^{\frac{1}{2}})$.

For example compute $Z=\langle\frac{2x_1}{1-t_1}e^{2t_1\tau_1+3t_3\tau_2+t_2\xi_1x_3+x_3y_3}\rangle_{1,2}$. At the first stage we contract along variables $\zeta=a_1,a_2,\tau_1,\tau_2$ by applying Theorem 1 with dual variables $z=(\alpha_1,\alpha_2,t_1,t_2)$ in that order, so $Q=L=\mathrm{Diag}(0,0,3,0)$, $\ell=(0,0,0,7t_3)^T$, $\lambda=0$ and the perturbation is $P(z,\zeta)=\frac{2x_1}{1-t_1}e^{t_2\xi_1x_3}$. This gives us $\tilde{Q}=\mathrm{Diag}(1,1,-\frac{1}{2},1)$ and so $-\frac{1}{2}e^{x_3y_3}\langle P(\tilde{Q}(z+\ell),\zeta)\rangle_{1,2}=-\frac{1}{2}e^{x_3y_3}\langle\frac{x_1}{1+\frac{1}{2}t_1}e^{(t_2+7t_3)\xi_1x_3}\rangle=-\frac{1}{2}x_1e^{x_3y_3+7\xi_1x_3}$ Finally the second stage is contraction along variables η_1,η_2,x_1,x_2 which in this case does not require much effort to get $Z=-\frac{7x_3}{2}e^{x_3y_3}$.

Since we may end up dividing by zero, not all contractions are well defined. Nevertheless we consider a partially defined composition rule as follows. Gaussians $f\in\mathcal{G}(I,J)$ and $g\in\mathcal{G}(J,K)$ may be composed as $g\circ f=\langle fg\rangle_J$ when defined this is an element of $\mathcal{G}(I,K)$. In the formula we assumed $I\cap K=\emptyset$, otherwise we temporarily rename the labels so they are disjoint, do the contraction and undo the renaming.

Definition 9. *By a subcategory $\tilde{\mathcal{G}}$ of two step Gaussians we mean a category $\tilde{\mathcal{G}}$ whose objects are finite sets, and for all I,J we have $\tilde{\mathcal{G}}(I,J)\subset\mathcal{G}(I,J)$ and composition is as above.*

2.3 The Gaussian tangle invariant

The main result of this paper is

Theorem 2. *There exists a functor $Z:\mathcal{R}\to\tilde{\mathcal{G}}$ from the category of rec-tangles $\mathcal{R}$ to a subcategory of two-step Gaussians $\tilde{\mathcal{G}}$. The functor is the identity on objects and sends union to product.*
Moreover $Z(X_{ij})=e^{b_ia_j}e_q^{y_ix_j}$, where $b_i=t_i+\epsilon a_i$ and $q=e^\epsilon$ and $e_q^z=\sum_{u=0}^\infty\frac{z^u}{[u]!}$ and $[u]!=\prod_{k=1}^u\frac{1-q^k}{1-q}$.

Up to the first order in ϵ the images of the basic rec-tangles under Z are, setting $T=e^{-t}, \mathcal{A}=e^\alpha$, the following:

$$Z(X_{ij})=e^{t_ia_j+y_ix_j}(1+\epsilon(a_ia_j-\frac{1}{4}y_i^2x_j^2)+\mathcal{O}(\epsilon^2)) \tag{1}$$

$$Z(X_{ij}^{-1})=e^{-t_ia_j-T_i^{-1}y_ix_j}(1+\epsilon((-a_ia_j-T_i^{-1}(a_i+a_j)y_ix_j-\frac{3}{4}T_i^{-2}y_i^2x_j^2)+\mathcal{O}(\epsilon^2)) \tag{2}$$

$$Z(m_k^{ij})=e^{(\alpha_i+\alpha_j)a_k+(\tau_i+\tau_j)t_k+(\eta_i+\eta_j\mathcal{A}_j^{-1})y_k+(\xi_i\mathcal{A}_j^{-1}+\xi_j)x_k+(1-T_k)\xi_j\eta_i}(1+\Lambda+\mathcal{O}(\epsilon^2)) \tag{3}$$

$$\Lambda=\xi_i\eta_j((2a_kT_k+y_ix_j\mathcal{A}_i^{-1}\mathcal{A}_j^{-1})+\frac{1}{2}(1-3T_k)(y_k\mathcal{A}_i^{-1}\eta_j+x_k\mathcal{A}_j^{-1}\xi_i)+\frac{\eta_j\xi_i}{4}(1-4T_k+3T_k^2)) \tag{4}$$

$$Z(\Delta_{\ell r}^i)=e^{(a_\ell+a_r)\alpha_i+(t_\ell+t_r)\tau_i+(y_\ell+y_rT_\ell)\eta_i+(x_\ell+x_r)\xi_i}(1+\epsilon(-a_\ell x_r\xi_i-T_\ell a_\ell y_r\eta_i+\frac{1}{2}T_\ell y_\ell y_r\eta_i^2+\frac{1}{2}x_\ell x_r\xi_i^2)+\mathcal{O}(\epsilon^2)) \tag{5}$$

$$Z(S_j^i)=e^{-\alpha_ia_j-\tau_it_j-T_j\mathcal{A}_i^{-1}\eta_iy_j-\mathcal{A}_i\xi_ix_j-(1-T_j^{-1})\mathcal{A}_i\eta_i\xi_i}(1+\epsilon\Sigma+\mathcal{O}(\epsilon^2)) \tag{6}$$

$$\Sigma=\frac{1}{2}\mathcal{A}_iT_j^{-1}(1-2a_j)\eta_i-\frac{1}{2}(1-2a_j)\mathcal{A}_ix_j\xi_i-\frac{1}{2}x_j^2\xi_i^2\mathcal{A}_i^2-\frac{1}{2T_j^2}y_j^2\mathcal{A}_i^2\eta_i^2 \tag{7}$$

$$+\frac{\xi_i\eta_i\mathcal{A}_i}{T_j}(2a_j-x_jy_j\mathcal{A}_i-1+T_j+\frac{1}{2}y_j\mathcal{A}(3T_j^{-1}-1)\eta_i+x_j\mathcal{A}_i(3-T_j)\xi_i+\frac{1}{4}\xi_i\eta_i\mathcal{A}_i(-3T_i^{-2}+4T_i^{-1}-1)) \tag{8}$$

$$Z(1_i)=Z(\epsilon^i)=1 \tag{9}$$

$$Z(C_i)=T^{\frac{1}{2}}e^{-\epsilon a_i}\qquad Z(\bar{C}_i)=T^{-\frac{1}{2}}e^{\epsilon a_i} \tag{10}$$

The 0-th order in ϵ determines the Alexander polynomial. The 1-st order in ϵ is able to distinguish all prime knots up to 10 crossings, outperforming the combination of HOMFLY polynomial and Khovanov homology. It essentially coincides with the polynomial time knot polynomial from our previous paper [**?**].

Computing up to order k in ϵ is possible in polynomial time in the number of crossings of the knot. A version of this statement also applies to tangles, see section Extras.

Our invariant is likely to be equivalent to the universal $U_q(\mathfrak{sl}_2)$ invariant and so is determined by and determines the colored Jones polynomials. More precisely we expect the n-th order in ϵ to be some combination of the n-loop approximations of the colored Jones polynomial. See also Rozansky and Overbay.

DO AN EXAMPLE WITH TREFOIL HERE ?

3 Construction of the invariant Z

3.1 Quasi-triangular categories

Definition 10. *(Quasi-triangular category) A category $\mathcal{C}$ whose objects are finite sets is called quasi-triangular if for any sets $I \cap I' = \emptyset = J \cap J'$ there exist maps $\cup : \mathcal{C}(I,J) \times \mathcal{C}(I',J') \to \mathcal{C}(I \cup I', J \cup J')$ and for any i,j,k there are morphisms $\mathrm{id}^i_j \in \mathcal{C}(i,j)$, $X_{i,j} \in \mathcal{C}(\emptyset, \{i,j\})$ and $m^{ij}_k \in \mathcal{C}(\{i,j\},\{k\})$ and $\Delta^i_{jk} \in \mathcal{C}(\{i\},\{jk\})$ and $S^i_j, \bar{S}^i_j \in \mathcal{C}(\{i\},\{j\})$ and $\epsilon^i \in \mathcal{C}(\{i\},\emptyset)$ and $1_i \in \mathcal{C}(\emptyset,\{i\})$ satisfying the following, (writing $\cup$ usually as juxtaposition):*

$$f \cup g = g \cup f, \quad f \cup (g \cup h) = f \cup (g \cup h), \quad (f \cup g) /\!\!/ h = (f /\!\!/ h) \cup g \tag{11}$$

$$\textit{for all morphisms f,g,h for which it makes sense,} \quad \mathrm{id}^i_j /\!\!/ f = f = f /\!\!/ \mathrm{id}^j_i \tag{12}$$

$$\Delta^i_{a1} /\!\!/ \Delta^1_{bc} = \Delta^i_{1b} /\!\!/ \Delta^1_{ab}, \quad m^{ab}_1 /\!\!/ m^{1c}_i = m^{bc}_1 /\!\!/ m^{a1}_i \tag{13}$$

$$m^{ij}_1 \Delta^1_{ab} = \Delta^i_{12} \Delta^j_{34} /\!\!/ m^{13}_a m^{24}_b \tag{14}$$

$$1_i /\!\!/ m^{ij}_k = 1_i /\!\!/ m^{ji}_k = \mathrm{id}^j_k \quad \Delta^i_{j1} /\!\!/ \epsilon^1 = \mathrm{id}^i_j = \Delta^i_{1j} /\!\!/ \epsilon^1 \tag{15}$$

$$\Delta^i_{12} /\!\!/ S^1_1 /\!\!/ m^{12}_k = \epsilon^i 1_k = \Delta^i_{12} /\!\!/ S^2_2 /\!\!/ m^{12}_k \tag{16}$$

$$S^i_j /\!\!/ \bar{S}^j_k = \mathrm{id}^i_k = \bar{S}^i_j /\!\!/ S^j_k \tag{17}$$

$$X_{1c} /\!\!/ \Delta^1_{ab} = X_{a2} X_{b1} /\!\!/ m^{12}_c, \quad X_{a1} /\!\!/ \Delta^1_{bc} = X_{1b} X_{2c} /\!\!/ m^{12}_a \tag{18}$$

$$\Delta^i_{12} X_{34} /\!\!/ m^{13}_j m^{24}_k = X_{12} \Delta^i_{34} /\!\!/ m^{14}_j m^{23}_k \tag{19}$$

$$X_{a1} /\!\!/ \epsilon^1 = 1_a = X_{1a} /\!\!/ \epsilon^1 \tag{20}$$

The relations in a quasi-triangular category encode those found in the tensor algebra of a quasi-triangular Hopf algebra, hence the name. Note that there is no notion of addition here, unlike in usual algebra. The graphical calculus employed here also agrees with that found in texts on quasi-triangular Hopf algebras.

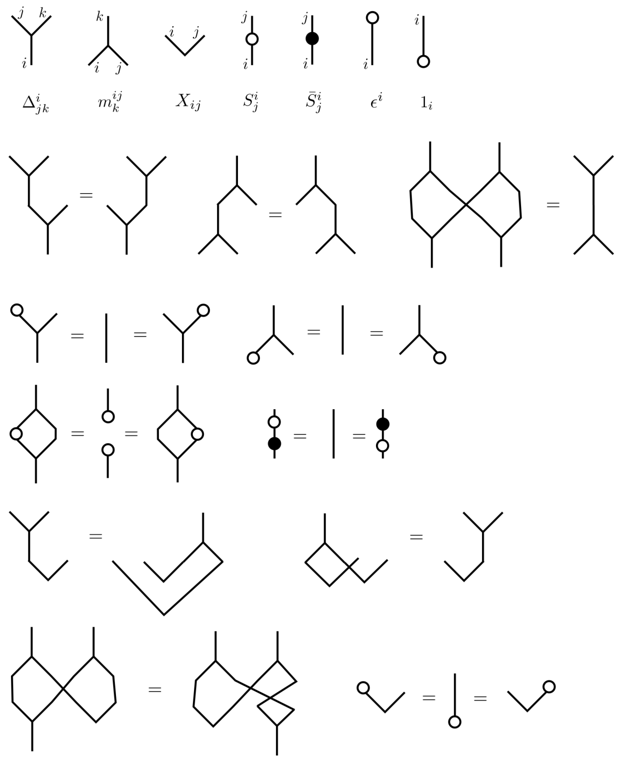

Figure 9: Operations and axioms in the quasi-triangular category $\mathcal{C}$.

The importance of quasi-triangular categories is the following lemma, whose proof can be found in the extras section.

Lemma 2. *There is a well-defined functor from the category $\mathcal{R}_0$ of rec-tangles with zero rotation number along the strands to any quasi-triangular category $\mathcal{C}$ sending the fundamental morphisms to the corresponding morphisms in $\mathcal{C}$ and sending union to union in $\mathcal{C}$.*

Below we will construct non-trivial examples of quasi-triangular categories using Gaussians. The tensor algebra of any quasi-triangular Hopf algebra produces additional if less practical examples of such categories.

Lemma 3. *The category of two-step Gaussians is quasi-triangular.*

Putting these two together proves:

Theorem 3. *There is a well-defined functor Z from the category $\mathcal{R}_0$ to the quasi-triangular category $\mathcal{C}$ characterized by $Z(f) = f$ for all structural morphisms f and sending union to union in $\mathcal{C}$.*

3.2 Extending to all rec-tangles

Next we explore how to extend our functor $Z : \mathcal{R}_0 \to \mathcal{C}$ to all of $\mathcal{R}$. First define the morphism w_i in $\mathcal{R}_0(\emptyset, \{i\})$ as shown in the picture. In fact $w_i = (X_{12}X_{34})/\!\!/S_1/\!\!/\bar{S}_3/\!\!/\bar{S}_3/\!\!/m_i^{1243}$

The element w satisfies some very natural equations that are readily seen using the language of rec-tangles. First set $\bar{w}_i = w_i /\!\!/ S_i$. We have

$$w_i/\!\!/\Delta^i_{\ell r} = w_\ell w_r \qquad w_1\bar{w}_2/\!\!/m_1^{12} = 1 \qquad w_1\bar{w}_3/\!\!/m_1^{123} = S_1/\!\!/S_1/\!\!/S_1/\!\!/S_1 \qquad w_1/\!\!/\epsilon^1 = 1$$

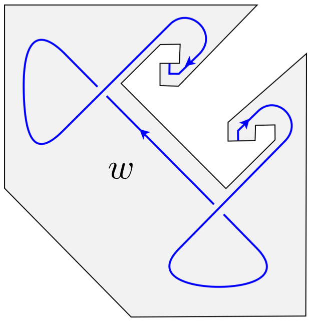

Figure 10: The rec-tangle w.

Lemma 4. *If there exists a 'square root' to w called $\bar{c}_i \in \mathcal{C}(\emptyset, \{i\})$ such that if we set $c_i = \bar{c}_i/\!\!/S_i$ we have*

$$Z(w_1) = \bar{c}_1\bar{c}_2/\!\!/m_1^{12} \tag{21}$$
$$\bar{c}_1/\!\!/\Delta^1_{\ell r} = \bar{c}_\ell\bar{c}_r \tag{22}$$
$$\bar{c}_1/\!\!/\epsilon^1 = 1 \tag{23}$$
$$c_1c_2/\!\!/m_1^{12} = 1 \tag{24}$$
$$\bar{c}_1c_3/\!\!/m_1^{123} = S_2/\!\!/S_2 \tag{25}$$

Then seting $Z(C_i) = \bar{c}_i/\!\!/S_i$ and $Z(\bar{C}_i) = \bar{c}_i$ extends the functor $Z : \mathcal{R}_0 \to \mathcal{C}$ to a functor $Z : \mathcal{R} \to \mathcal{C}$.

Proof. ???? □

Concatenating C on both sides of w shows that $1 = Z(1_1) = Z(C_1w_2C_3/\!\!/m_1^{123})$. In other words, the values of the two positive kinks of rotation numbers ± 1 are equal. The same goes for the two negative kinks. This fact is an important motivation for the above element w.

We will show that there exists $\bar{c}$ in the two step Gaussians satisfying the above conditions.

3.3 A double construction for categories

In this section we show how one may construct quasi-triangular categories from simpler categories we call purple categories. This construction is a version of Drinfeld's quantum double construction for quasi-triangular Hopf algebras. Our point of view emphasizes the role of the crossing (R-matrix) and the pairing, putting them on equal footing. From a knot theory perspective this seems most natural. In the Gaussian examples below it is also most convenient.

Definition 11. *A purple category is a category whose objects are finite sets and includes the following morphisms. $(m_{\mathbb{A}})^{ij}_k, (m_{\mathbb{B}})^{ij}_k \in \mathcal{P}(\{i,j\},\{k\})$ for any $i \neq j$ and also $(1_{\mathbb{A}})_i, (1_{\mathbb{B}})_i \in \mathcal{P}(\emptyset, \{i\})$. Next there are $J^{ij}_k \in \mathcal{P}(\{i,j\},\{k\})$ and $\mathrm{DI}^i_{jk} \in \mathcal{P}(\{i\},\{j,k\})$, joining and divorcing the $\mathbb{A}$ and $\mathbb{B}$ parts. Finally we have positive/negative crossings $X_{i,j}, \bar{X}_{i,j} \in \mathcal{P}(\emptyset, \{i,j\})$ and pairings $P_{i,j}, \bar{P}_{i,j} \in \mathcal{P}(\{i,j\}, \emptyset)$.*

Also there are natural union maps $\cup : \mathcal{P}(I,J) \times \mathcal{P}(I',J') \to \mathcal{P}(I \cup I', J \cup J')$ as long as $I \cap I' = \emptyset = J \cap J'$. The union should be associative in that when applied to disjoint triples the order is unimportant.

Finally for any I, J there is a morphism $\dagger \in \mathcal{P}(I,J)$ satisfying $\dagger/\!\!/f = \dagger = f/\!\!/\dagger = f \cup \dagger$ for any f.

Furthermore the morphisms should satisfy the following axioms, see figure 11 for the graphical version.

$$\mathrm{DI}^i_{j2}(1_{\mathbb{B}})_3/\!\!/P^{23} = (\mathrm{id}_{\mathbb{B}})^i_j \qquad \mathrm{DI}^i_{3j}(1_{\mathbb{A}})_2/\!\!/P^{23} = (\mathrm{id}_{\mathbb{A}})^i_j \tag{26}$$

$$\mathrm{DI}^i_{12}/\!\!/J^{12}_j = (\mathrm{id}_{\mathbb{A}})^i_j(\mathrm{id}_{\mathbb{B}})^i_j \qquad J^{ij}_1/\!\!/\mathrm{DI}^1_{k\ell} = (\mathrm{id}_{\mathbb{A}})^i_k(\mathrm{id}_{\mathbb{B}})^j_\ell \tag{27}$$

$$(\mathrm{id}_{\mathbb{A}})^i_1/\!\!/\mathrm{DI}^1_{jk} = (1_{\mathbb{B}})_j(\mathrm{id}_{\mathbb{A}})^i_k \qquad (\mathrm{id}_{\mathbb{B}})^i_1/\!\!/\mathrm{DI}^1_{jk} = (1_{\mathbb{B}})_k(\mathrm{id}_{\mathbb{A}})^i_j \tag{28}$$

$$m^{ij}_1/\!\!/m^{1k}_\ell = m^{jk}_1/\!\!/m^{i1}_\ell \qquad 1_i/\!\!/m^{ij}_k = \mathrm{id}^k_j = 1_i/\!\!/m^{ji}_k \tag{29}$$

$$X_{12}\bar{X}_{34}/\!\!/(m_{\mathbb{B}})^{13}_k(m_{\mathbb{A}})^{24}_\ell = (1_{\mathbb{B}})_k(1_{\mathbb{A}})_\ell = \bar{X}_{12}X_{34}/\!\!/(m_{\mathbb{B}})^{13}_k(m_{\mathbb{A}})^{24}_\ell \tag{30}$$

$$\bar{X}_{j1}/\!\!/\bar{P}^{1i} = (\mathrm{id}_{\mathbb{B}})^i_j = X_{j1}/\!\!/P^{1i} \qquad \bar{X}_{1j}/\!\!/\bar{P}^{i1} = (\mathrm{id}_{\mathbb{A}})^i_j = X_{1j}/\!\!/P^{i1} \tag{31}$$

$$(m_{\mathbb{A}})^{ij}_2(1_{\mathbb{B}})_3/\!\!/P^{23} = (1_{\mathbb{B}})_2(1_{\mathbb{B}})_3/\!\!/P^{i2}P^{j3} \qquad (m_{\mathbb{B}})^{ij}_3(1_{\mathbb{A}})_2/\!\!/P^{23} = (1_{\mathbb{A}})_2(1_{\mathbb{A}})_3/\!\!/P^{2i}P^{3j} \tag{32}$$

$$X_{i1}X_{j2}X_{3k}X_{4\ell}/\!\!/(m_{\mathbb{A}})^{21}_5(m_{\mathbb{B}})^{43}_6/\!\!/P^{56} = X_{12}X_{36}X_{54}X_{78}/\!\!/(m_{\mathbb{B}})^{13}_i(m_{\mathbb{B}})^{57}_j(m_{\mathbb{A}})^{86}_k(m_{\mathbb{A}})^{42}_\ell \tag{33}$$

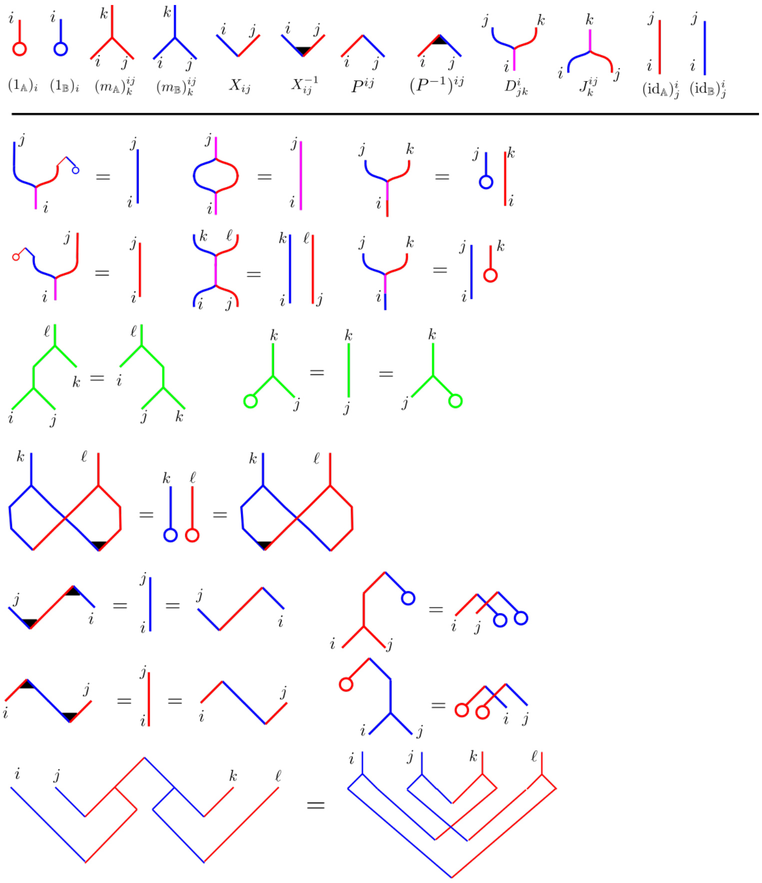

Figure 11: Operations and axioms in the purple category $\mathcal{P}$. Red stands for $\mathbb{A}$, blue for $\mathbb{B}$ and green means either all blue or all red.

The color coding of the strands is Blue for $\mathbb{B}$ and red (Adom in Hebrew) for $\mathbb{A}$. Generally we are dealing with a mix of both $\mathbb{A}$ and $\mathbb{B}$ and this is drawn in purple hence the category's name. Notice that the D map splits a purple line into its red and blue constituents. The other structural maps all act on either red or blue meaning that they will send everything else to 0. The color green means either all blue or all red.

Some other axioms describe the associativity of $m_{\mathbb{A}}, m_{\mathbb{B}}$ and the properties of the unit. Then there are axioms defining the multiplicative inverse of X and P as a partial compositional inverse. Finally the axiom at the bottom is most intriguing and is hardest to verify in our examples. We will see later that it is a restatement of the multiplicativity of the two coproducts in the sense of the next definition.

Define the following composite morphisms see figure 12 used in the next theorem.

Definition 12.

$$(\Delta_{\mathbb{B}})^i_{jk} = X_{j1}X_{k2}/\!\!/(m_{\mathbb{A}})^{12}_3/\!\!/P^{3i} \qquad (S_{\mathbb{B}})^i_j = \bar{X}_{j1}/\!\!/P^{1i} \qquad (\bar{S}_{\mathbb{B}})^i_j = X_{j1}/\!\!/\bar{P}^{1i} \qquad \epsilon^i_{\mathbb{B}} = (1_{\mathbb{A}})_j/\!\!/P^{ji} \tag{34}$$

$$(\Delta_{\mathbb{A}})^i_{jk} = X_{1j}X_{2k}/\!\!/(m_{\mathbb{B}})^{12}_3/\!\!/P^{i3} \qquad (S_{\mathbb{A}})^i_j = \bar{X}_{1j}/\!\!/P^{i1} \qquad (\bar{S}_{\mathbb{A}})^i_j = X_{1j}/\!\!/\bar{P}^{i1} \qquad \epsilon^i_{\mathbb{A}} = (1_{\mathbb{B}})_j/\!\!/P^{ij} \tag{35}$$

$$\mu^{ij}_{k\ell} = (\Delta_{\mathbb{A}})^i_{12}/\!\!/(\Delta_{\mathbb{A}})^2_{23}(\Delta_{\mathbb{B}})^j_{45}/\!\!/(\Delta_{\mathbb{B}})^5_{56}/\!\!/P^{16}\bar{P}^{34} \tag{36}$$

With the definition of Δ the final axiom of purple categories is equivalent to the more familiar yet less symmetric

$$(m_{\mathbb{A}})^{\ell m}_I/\!\!/(\Delta_{\mathbb{A}})^I_{jk} = ((\Delta_{\mathbb{A}})^\ell_{ab}(\Delta_{\mathbb{A}})^m_{cd})/\!\!/((m_{\mathbb{A}})^{ac}_j(m_{\mathbb{A}})^{bd}_k)$$

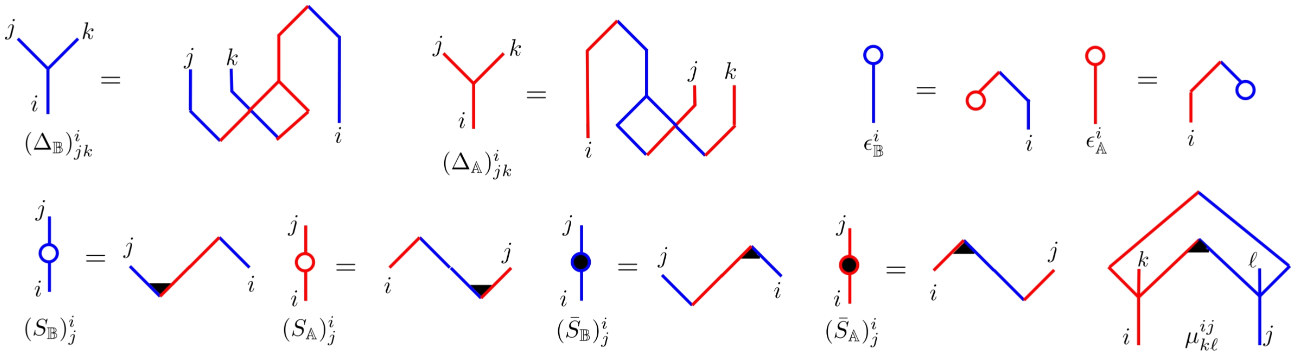

Figure 12: Defining coproducts Δ, counits ϵ and antipodes $S, \bar{S}$ in terms of the fundamental morphisms $X, P, m, 1$.

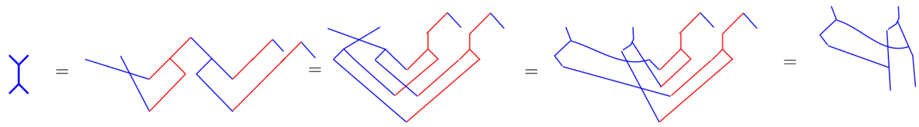

Figure 13: Operations in the Drinfeld double.

and also equivalent with the same for $\mathbb{B}$.

Theorem 4. *(Double construction)*
A purple category becomes a quasi-triangular category if we set, see figure 14

$$m^{ij}_k = (\mathrm{DI}^i_{i_1 i_2}\mathrm{DI}^j_{j_1 j_2})/\!/\mu^{i_2 j_1}_{fg}/\!/(m_{\mathbb{B}})^{i_1 g}_{k_1}(m_{\mathbb{A}})^{f j_2}_{k_2}/\!/J^{k_1 k_2}_k \quad \epsilon^i = \mathrm{DI}^i_{jk}/\!/\epsilon^j_{\mathbb{A}}\epsilon^k_{\mathbb{B}} \quad 1_i = (1_{\mathbb{A}})_{i_1}(1_{\mathbb{B}})_{i_2}/\!/J^{i_1 i_2}_i \tag{37}$$

$$\Delta^i_{jk} = \mathrm{DI}^i_{rs}/\!/((\Delta_{\mathbb{B}})^r_{k_2 j_1}(\Delta_{\mathbb{A}})^s_{j_2 k_2})/\!/J^{j_1 j_2}_j J^{k_1 k_2}_k \quad S^i_j = \mathrm{DI}^i_{jk}/\!/((\bar{S}_{\mathbb{B}})^j_1(S_{\mathbb{A}})^k_2)/\!/m^{21}_j \quad \bar{S}^i_j = \mathrm{DI}^i_{jk}/\!/((S_{\mathbb{B}})^j_1(\bar{S}_{\mathbb{A}})^k_2)/\!/m^{21}_j \tag{38}$$

Proof. PICTURES IN THIS PROOF ARE ALL MIRRORED DUE TO OLD AXBY CONVENTIONS NEEDS REVISION. Since the formulas get long and cluttered we prefer to carry out the proof pictorially. The proof follows the usual one in the Drinfeld double construction found in Majid or Etingof-Schiffman but since our conventions are non-standard we present it anyway. We start by checking the associativity and unit axioms of m^{ij}_k, see figure 15 and figure 16.

Next we check the algebra morphism property. Finally the convolution inverse property and the two quasi-triangular axioms.

□

Our task now is to provide examples of purple categories.

3.4 Gaussian meta Hopf algebras

3.4.1 Toy example: one step Gaussians

Our first example of a purple category is the category $\mathcal{G}_1$ with objects finite sets and morphisms given by two variable Gaussians. The variables are x, y with duals ξ, η. Define for finite sets I, J the morphisms $\mathcal{G}_1(I,J) = \{e^{(yx)Q\binom{\eta}{\xi}}\}$ where Q is a matrix with coefficients in $\mathbb{Q}$ and $x = (x_j)_{j\in J}, y = (y_j)_{j\in J}$ and $\eta = (\eta_i)_{i\in I}, \xi = (\xi_i)_{i\in I}$.

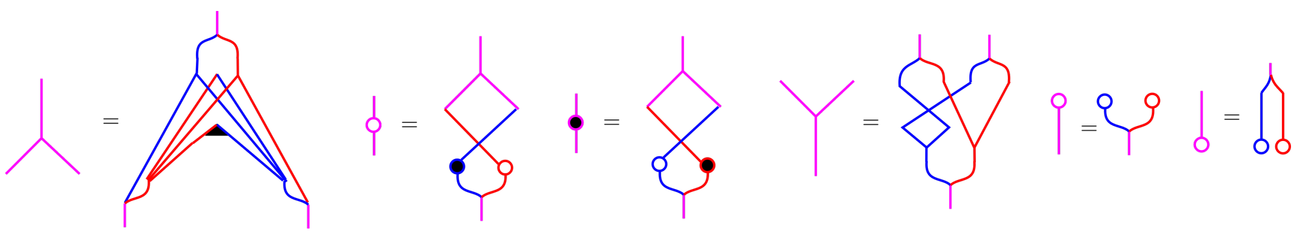

Figure 14: Operations in the Drinfeld double.

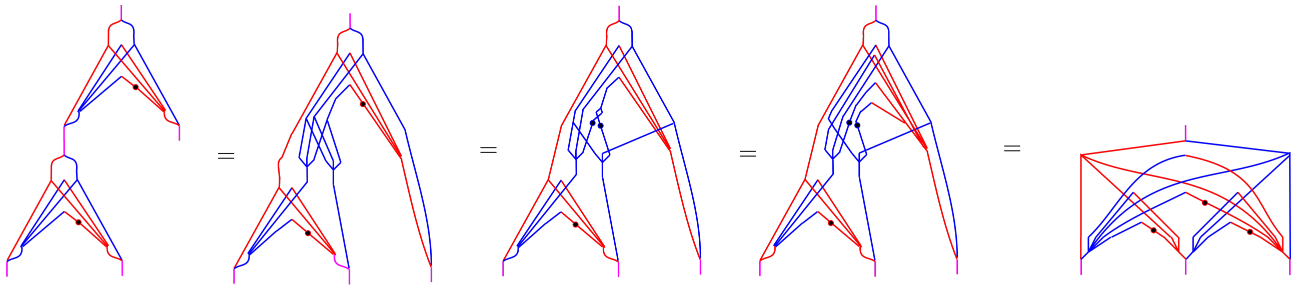

Figure 15: Associativity of m_k^{ij}

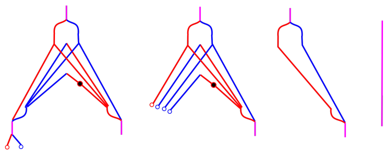

Figure 16: Unit axiom of m_k^{ij}

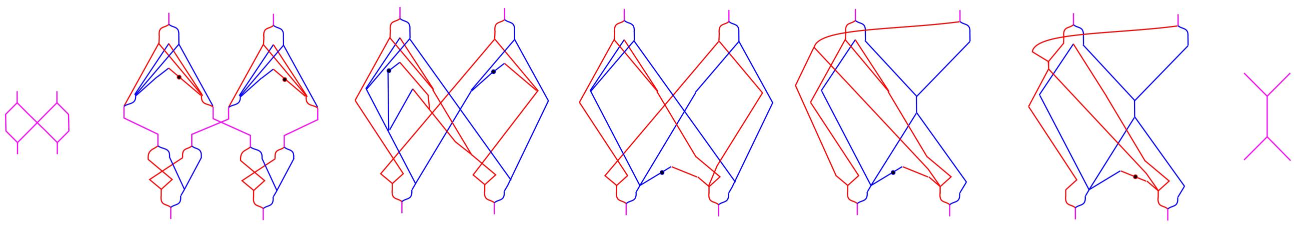

Figure 17: Algebra morphism axiom

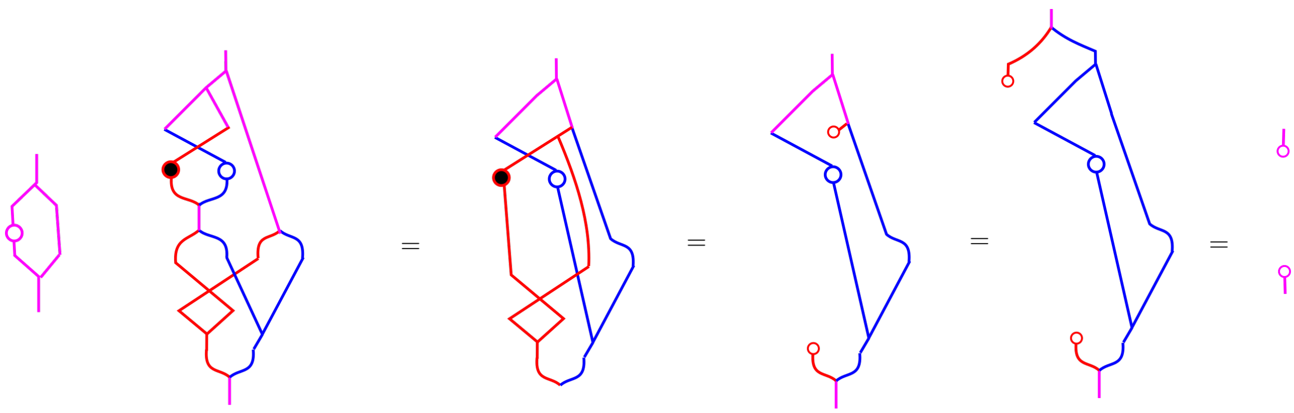

Figure 18: convolution inverse property of the antipode

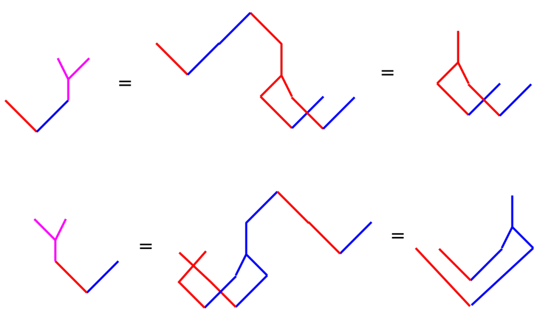

Figure 19: First quasi-triangular axioms

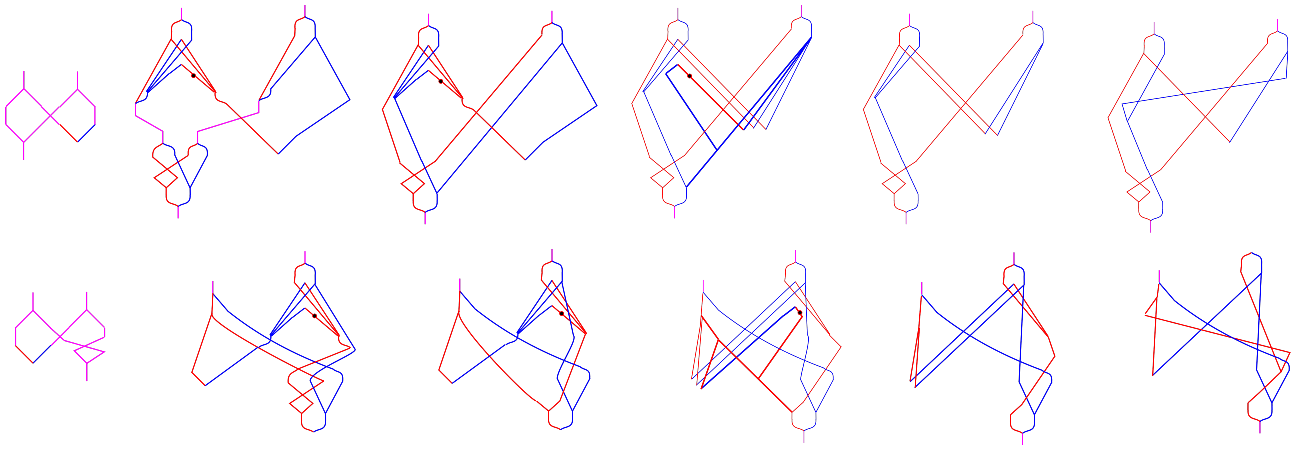

Figure 20: Second quasi-triangular axioms. First row deals with the red input, second row with the blue part of the Delta input in a similar (dual) fashion.

Composition of morphisms is defined as follows. For $g \in \mathcal{G}_1(I,J)$ and $f \in \mathcal{G}_1(J,K)$ we set $g /\!\!/ f = f \circ g = \langle fg \rangle_{\{x_j, y_j | j \in J\}}$. If the contraction is not well-defined we artificially set it to be †. Union of morphisms in $\mathcal{G}_1$ is given by multiplication: $f \cup g = fg$.

Next we define the structural morphisms:

$$(m_{\mathbb{A}})^{ij}_k = e^{(\xi_i+\xi_j)x_k} \qquad (m_{\mathbb{B}})^{ij}_k = e^{(\eta_i+\eta_j)y_k} \quad (1_{\mathbb{A}})_i = (1_{\mathbb{B}})_i = 1 \tag{39}$$

$$\mathrm{DI}^i_{jk} = e^{\xi_i x_k + \eta_i y_j} \qquad X_{ij} = e^{y_i x_i} \quad J^{ij}_k = e^{(\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k} \tag{40}$$

Inverting naturally gives us the other formulas:

$$\bar{X}_{ij} = e^{-y_i x_j} \quad P^{ij} = e^{\xi_i \eta_j} \quad \bar{P}^{ij} = e^{-\xi_i \eta_j} \tag{41}$$

And it follows from the definitions that

$$(\Delta_{\mathbb{A}})^i_{\ell r} = X_{1\ell} X_{2r} /\!\!/ (m_{\mathbb{B}})^{12}_3 /\!\!/ P^{i3} = \langle e^{y_1 x_\ell + y_2 x_r + (\eta_1+\eta_2)y_3 + \xi_i \eta_3} \rangle_{1,2,3} = e^{\xi_i(x_\ell + x_r)} \tag{42}$$

$$(\Delta_{\mathbb{B}})^i_{\ell r} = X_{\ell 1} X_{r2} /\!\!/ (m_{\mathbb{A}})^{12}_3 /\!\!/ P^{3i} = \langle e^{y_\ell x_1 + y_r x_2 + (\xi_1+\xi_2)x_3 + \xi_3 \eta_i} \rangle_{1,2,3} = e^{\eta_i(y_\ell + y_r)} \tag{43}$$

$$(S_{\mathbb{A}})^i_j = \bar{X}_{1j} /\!\!/ P^{i1} = \langle e^{-y_1 x_j + \xi_i \eta_1} \rangle_1 = e^{-\xi_i x_j} \tag{44}$$

$$(\bar{S}_{\mathbb{A}})^i_j = X_{1j} /\!\!/ \bar{P}^{i1} = \langle e^{y_1 x_j - \xi_i \eta_1} \rangle_1 = e^{-\xi_i x_j} \tag{45}$$

$$(S_{\mathbb{B}})^i_j = \bar{X}_{j1} /\!\!/ P^{1i} = \langle e^{-y_j x_1 + \xi_1 \eta_i} \rangle_1 = e^{-\eta_i y_j} \tag{46}$$

$$(\bar{S}_{\mathbb{B}})^i_j = X_{j1} /\!\!/ \bar{P}^{1i} = \langle e^{y_j x_1 - \xi_1 \eta_i} \rangle_1 = e^{-\eta_i y_j} \tag{47}$$

$$\epsilon^i_{\mathbb{A}} = (1_{\mathbb{B}})_j /\!\!/ P^{ij} = \langle e^{\xi_i \eta_j} \rangle_j = 1 \tag{48}$$

$$\epsilon^i_{\mathbb{B}} = (1_{\mathbb{A}})_j /\!\!/ P^{ji} = \langle e^{\xi_j \eta_i} \rangle_j = 1 \tag{49}$$

$$\tag{50}$$

$$\mu^{ij}_{k\ell} = (\Delta_{\mathbb{A}})^i_{12} /\!\!/ (\Delta_{\mathbb{A}})^2_{23} (\Delta_{\mathbb{B}})^j_{45} /\!\!/ (\Delta_{\mathbb{B}})^5_{56} /\!\!/ P^{16} \bar{P}^{34} = \langle e^{\xi_i(x_1+x_k+x_3) + \eta_j(y_4+y_\ell+y_6) + \xi_1\eta_6 - \xi_3\eta_4} \rangle_{1,3,4,6} =$$
$$\langle e^{\xi_i(\eta_6 + x_k - \eta_4) + \eta_j(y_4+y_\ell+y_6)} \rangle_{4,6} = e^{x_k \xi_i + \eta_j y_\ell + \xi_i \eta_j - \xi_i \eta_j} = e^{\xi_i x_k + \eta_j y_\ell}$$

To prove that we have a purple category we need to verify the multiplicativity of the coproduct axiom for either $\mathbb{B}$ or equivalently the $\mathbb{A}$ side. All other verifications are similar but easier.

$$(m_{\mathbb{A}})^{\ell m}_1 /\!\!/ (\Delta_{\mathbb{A}})^1_{jk} = (\Delta_{\mathbb{A}})^\ell_{ab} (\Delta_{\mathbb{A}})^m_{cd}) /\!\!/ ((m_{\mathbb{A}})^{ac}_j (m_{\mathbb{A}})^{bd}_k$$

The left hand side becomes

$$\langle e^{(\xi_\ell+\xi_m)x_1 + \xi_1(x_j+x_k))} \rangle_1 = e^{(\xi_\ell+\xi_m)(x_j+x_k)}$$

The right hand side is

$$\langle e^{\xi_\ell(x_a+x_b) + \xi_m(x_c+x_d) + (\xi_a+\xi_c)x_j + (\xi_b+\xi_d)x_k} \rangle_{a,b,c,d} = e^{\xi_\ell(x_j+x_k) + \xi_m(x_j+x_k)}$$

Now that we verified that we get a purple category we can use the doubling theorem to produce an example of quasi-triangular category. In addition we note that † does not really turn up in any combination we actually use:

Lemma 5. *$\dagger$ is not in the subcategory of $\mathcal{G}_1$ generated by unions of the morphisms D, J, μ, $\Delta_{\mathbb{A}}$, $\Delta_{\mathbb{B}}$, $m_{\mathbb{A}}$, $m_{\mathbb{B}}$, $1_{\mathbb{A}}$, $1_{\mathbb{B}}$, $S_{\mathbb{A}}$, $\bar{S}_{\mathbb{A}}$, $S_{\mathbb{B}}$, $\bar{S}_{\mathbb{B}}$, X, $\bar{X}$, $\epsilon_{\mathbb{A}}$, $\epsilon_{\mathbb{B}}$.*

Proof. To compute such combinations we just have to contract the product of the building blocks. Notice the matrix Q becomes an upper triangular block matrix if we order variables x, η and y, ξ. Terms $\eta\xi$ do not appear and neither do $x_i\xi_i$ or $\eta_i y_i$. That means the determinant of $I - Q$ must be non-zero and so the composition is not equal to $\dagger$. □

For example the multiplication comes out as:

$$m^{ij}_k = (\mathrm{DI}^i_{i_1 i_2}\mathrm{DI}^j_{j_1 j_2}) /\!\!/ \mu^{i_2 j_1}_{fg} /\!\!/ (m_{\mathbb{B}})^{i_1 g}_{k_1}(m_{\mathbb{A}})^{f j_2}_{k_2} /\!\!/ J^{k_1 k_2}_k = \tag{51}$$

$$\langle e^{\xi_i x_{i_2} + \eta_i y_{i_1} + \xi_j x_{j_2} + \eta_j y_{j_1} + \xi_{i_2} x_f + \eta_{j_1} y_g + (\eta_{i_1} + \eta_g) y_k + (\xi_f + \xi_{j_2}) x_k} \rangle_{f,g,i_1,i_2,j_1,j_2} = J^{ij}_k \tag{52}$$

Also, the element w used for extending to all rec-tangles can be computed easily using $S_i /\!\!/ S_i = \mathrm{id}_i$. We find $w_i =$

$$(X_{12}X_{34}) /\!\!/ S_1 /\!\!/ \bar{S}_3 /\!\!/ \bar{S}_3 /\!\!/ m^{1243}_i = \langle e^{y_1 x_2 + y_3 x_4 - \eta_1 y_{1'} - \xi_1 x_{1'} + (\xi_{1'} + \xi_2 + \xi_4 + \xi_3) x_i + (\eta_{1'} + \eta_2 + \eta_4 + \eta_3) y_i} \rangle_{1,2,3,4,1'}$$

$$= \langle e^{-y_1 x_2 + y_3 x_4 + (\xi_1 + \xi_2 + \xi_4 + \xi_3) x_i + (\eta_1 + \eta_2 + \eta_4 + \eta_3) y_i} \rangle_{3,4,2,1} = e^{y_i x_i - y_i x_i} = 1$$

Hence we may take $c_i = \bar{c}_i = 1$ to extend our invariant in a rather trivial way with $Z(C_i) = 1$.

3.4.2 Main example: Two step Gaussians

Augmenting the sets $\mathcal{G}(I, J)$ of two step docile perturbed Gaussians with index sets I, J with the morphism $\dagger$ shows that $\mathcal{G}$ becomes a category with respect to the usual composition (contraction of union). The aim of this section is to show that in fact the category $\mathcal{G}$ is purple with respect to the following choices:

Set $\mathcal{A} = e^{\alpha}, \mathcal{B} = e^{\epsilon\beta}$ and define the fundamental morphisms of the purple category to be:

$$\mathrm{DI}^i_{jk} = e^{\alpha_i(a_j + a_k) + \beta_i(b_j + b_k) + \xi_i(x_j + x_k) + \eta_i(y_j + y_k)} \tag{53}$$

$$J^{ij}_k = e^{(\alpha_i + \alpha_j) a_k + (\beta_i + \beta_j) b_k + (\xi_i + \xi_j) x_k + (\eta_i + \eta_j) y_k} \tag{54}$$

$$(\mathrm{id}_{\mathbb{A}})^i_j = e^{\alpha_i a_j + \xi_i x_j} \tag{55}$$

$$(\mathrm{id}_{\mathbb{B}})^i_j = e^{\beta_i b_j + \eta_i y_j} \tag{56}$$

$$(1_{\mathbb{A}})_i = (1_{\mathbb{B}})_i = 1 \tag{57}$$

$$(m_{\mathbb{A}})^{ij}_k = e^{(\alpha_i + \alpha_j) a_k + (\mathcal{A}_j^{-1}\xi_i + \xi_j) x_k} \tag{58}$$

$$(m_{\mathbb{B}})^{ij}_k = e^{(\beta_i + \beta_j) b_k + (\eta_i + \mathcal{B}_i^{-1}\eta_j) y_k} \tag{59}$$

$$X_{ij} = e^{b_i a_j} e_q^{y_i x_j} \tag{60}$$

$$P^{ij} = e^{\alpha_i \beta_j} E_q^{\xi_i \eta_j} \tag{61}$$

Here $q = e^{-\epsilon}$ and $e^z_q = \sum_{u=0}^{\infty} \frac{z^u}{[u]!}$ and $[u]! = \prod_{k=1}^{u} \frac{1-q^k}{1-q}$. Also set $E^z_q = \sum_{u=0}^{\infty} \frac{z^u [u]!}{u! u!}$.

The next lemma shows that all these series, in particular X_{ij} are in fact docile. Recall the q-exponential $e^z_q = \sum_n \frac{z^n}{[n]!}$, see lemma 6.

Lemma 6. *(Faddeev, Zagier, Quesne)*

$$e^z_q = \exp \sum_n \frac{(1-q)^n z^n}{n(1-q^n)}$$

Hence it and X_{ij} are docile perturbed two step Gaussian.

Also $\bar{X}_{ij} = e^{-b_i a_j - \frac{y_i x_j}{\mathcal{B}_i}}(1 + \mathcal{O}(\epsilon))$ and $\bar{P}^{ij} = e^{-\alpha_i \beta_j - \mathcal{A}_i \xi_i \eta_j}(1 + \mathcal{O}(\epsilon))$ and $P^{ij} = e^{\alpha_i \beta_j + \xi_i \eta_j}(1 + \mathcal{O}(\epsilon))$ are defined uniquely as power series by their relation to X and are docile perturbed two step Gaussians. Moreover P agrees with the formula given above.

Proof. Following Zagier (add reference), since the q-exponential is equal to its q-derivative we have $e^z_q = \frac{e^{qz}_q - e^z_q}{qz - z}$ so $\log e^{qz}_q = \log(1 + (1-q)z) + \log e^z_q$. Writing $\log e^z_q = \sum_n c_n z^n$ the previous implies that $q^n c_n = -\frac{(1-q)^n}{n} + c_n$ proving the formula.

If we set $\bar{X}_{ij} = e^Q P$ with $P = \sum_k P_k \epsilon^k$ our task is to compute P_k in terms of Q and $P_1, \ldots P_{k-1}$ and to estimate its degree. Our argument is by induction on k so assume the P_i for $i < k$ are defined uniquely and are of degree $\leq 4k$. Substituting into the defining equation $X_{ij}\bar{X}_{rs} /\!\!/ (m_{\mathbb{B}})^{ir}_1 (m_{\mathbb{A}})^{js}_2 = 1$ we get

$$X_{ij}(e^Q \sum_{\ell < k} P_\ell \epsilon^\ell)_{rs} /\!\!/ (m_{\mathbb{B}})^{ir}_1 (m_{\mathbb{A}})^{js}_2 + X_{ij}(e^Q P_k \epsilon^k)_{rs}) /\!\!/ (m_{\mathbb{B}})^{ir}_1 (m_{\mathbb{A}})^{js}_2 = 1 + \mathcal{O}(\epsilon^{k+1})$$

and hence after cancelling X_{ij} to 0-th order we get

$$(e^Q P_0)_{gh} X_{ij} (e^Q \sum_{\ell<k} P_\ell \epsilon^\ell)_{rs} /\!\!/ (m_{\mathbb{B}})^{gi}_i (m_{\mathbb{A}})^{hj}_j /\!\!/ (m_{\mathbb{B}})^{ir}_1 (m_{\mathbb{A}})^{js}_2 + (e^Q P_k \epsilon^k)_{12} = 1 + \mathcal{O}(\epsilon^{k+1})$$

Comparing the coefficient of ϵ^k we see that P_k is determined uniquely and its degree must be $\leq 4k$ since X is docile.

Using the defining equation $X_{ij} /\!\!/ P^{jk} = (\mathrm{id}_{\mathbb{B}})^i_k$ we similarly conclude that there is a unique docile Gaussian expression for P^{ij}. Likewise the defining equation $\bar{X}_{ij} /\!\!/ \bar{P}^{jk} = (id_{\mathbb{B}})^i_k$ in the same way tells us there is a unique docile Gaussian for $\bar{P}$.

Finally we need to check that the formula for P given above satifies the defining equation $X_{ij} /\!\!/ P^{jk} = (\mathrm{id}_{\mathbb{A}})^i_k$. By uniqueness this will prove that this formula gives a docile Gaussian.

$$X_{ij} /\!\!/ P^{jk} = \langle e^{b_i a_j + \alpha_j \beta_k} \sum_{u,v} \frac{(y_i x_j)^u (\xi_j \eta_k)^v [v]!}{[u]!(v!)^2} \rangle_j = e^{\beta_k b_i} \sum_u \frac{(y_i \eta_k)^u}{u!} = e^{\beta_k b_i + \eta_k y_i} = (\mathrm{id}_{\mathbb{B}})^i_k$$

□

Next we check the essential property of purple categories:

Lemma 7. $\Delta_{\mathbb{A}}$ *has the algebra morphism property*

$$(m_{\mathbb{A}})^{ij}_1 /\!\!/ (\Delta_{\mathbb{A}})^1_{\ell r} = (\Delta_{\mathbb{A}})^i_{12} (\Delta_{\mathbb{A}})^j_{34} /\!\!/ (m_{\mathbb{A}})^{13}_\ell (m_{\mathbb{A}})^{24}_r$$

Proof. We start by computing the effect of $\Delta_{\mathbb{A}}$ on linear expressions.

$$(\Delta_{\mathbb{A}})^i_{\ell r} = X_{1\ell} X_{2r} /\!\!/ (m_{\mathbb{B}})^{12}_3 /\!\!/ P^{i3} = \langle e^{b_1 a_\ell + b_2 a_r + (\beta_1+\beta_2) b_3 + (\eta_1 + \mathcal{B}_1^{-1} \eta_2) y_3} e_q^{y_1 x_\ell} e_q^{y_2 x_r} \rangle_{12} /\!\!/ P^{i3} =$$

$$e^{(a_\ell + a_r) b_3} e_q^{y_3 x_\ell} e_q^{e^{-\epsilon a_\ell} y_3 x_r} /\!\!/ P^{i3} = \langle e^{\alpha_i \beta_3 + (a_\ell + a_r) b_3} e_q^{y_3 x_\ell} e_q^{e^{-\epsilon a_\ell} y_3 x_r} E_q^{\xi_i \eta_3} \rangle_3 = \langle e^{(a_\ell + a_r)\alpha_i} \sum_{k,l,m} (y_3 x_\ell)^k (e^{-\epsilon a_\ell} y_3 x_r)^l (\xi_i \eta_3)^m \frac{[m]!}{[k]![l]! m!^2} \rangle$$

$$= e^{(a_\ell + a_r)\alpha_i} \sum_{k,l} x_\ell^k e^{-\epsilon a_\ell l} x_r^l \xi_i^{k+l} \frac{[k+l]!}{[k]![l]!(k+l)!}$$

And so we have $a_i /\!\!/ (\Delta_{\mathbb{A}})^i_{\ell r} = a_\ell + a_r$ and $x_i /\!\!/ (\Delta_{\mathbb{A}})^i_{\ell r} = x_\ell + e^{-\epsilon a_l} x_r$ and of course $b_i /\!\!/ (\Delta_{\mathbb{A}})^i_{\ell r} = y_i /\!\!/ (\Delta_{\mathbb{A}})^i_{\ell r} = 0$.

Next, introduce a new temporary morphism ∇ that agrees with $\Delta_{\mathbb{A}}$ on the linear expressions and extend it to general expressions by assuming that it has the sought after algebra morphism property:

$$(m_{\mathbb{A}})^{ij}_1 /\!\!/ \nabla^1_{\ell r} = \nabla^i_{12} \nabla^j_{34} /\!\!/ (m_{\mathbb{A}})^{13}_\ell (m_{\mathbb{A}})^{24}_r$$

that also agrees with $\Delta_{\mathbb{A}}$ on the linear expressions so This does actually define a ∇ uniquely and we will may derive a formula for ∇ below. By inspection this formula agrees with that for $\Delta_{\mathbb{A}}$ and so the proof will be finished.

The formula for ∇ comes from expanding the trivial identity:

$$\nabla^i_{\ell,r} = (\mathrm{id}_{\mathbb{A}})^i_i /\!\!/ \nabla^i_{\ell r} = \sum_{m,n} \frac{(\alpha_i a_i)^m (\xi_i x_i)^n}{m!n!} /\!\!/ \nabla^i_{\ell r}$$

Since $a_i^m = a_1^{m-1} a_2 /\!\!/ (m_{\mathbb{A}})^{12}_i$ it follows from induction and the above property of ∇ that $a_i^m /\!\!/ \nabla^i_{\ell r} = (a_\ell + a_r)^m$. Likewise $x_i^n = x_1^{n-1} x_2 /\!\!/ (m_{\mathbb{A}})^{12}_i$ means

$$x_i^n /\!\!/ \nabla^i_{\ell r} = (x_{1'}^{n-1} /\!\!/ \nabla^{1'}_{12})(x_3 + e^{-\epsilon a_3} x_4) /\!\!/ (m_{\mathbb{A}})^{13}_\ell (m_{\mathbb{A}})^{24}_r =$$

$$\langle (x_{1'}^{n-1} /\!\!/ \nabla^{1'}_{12})(x_3 + e^{-\epsilon a_3} x_4) e^{(\alpha_1+\alpha_3) a_\ell + (\alpha_2+\alpha_4) a_r + (\mathcal{A}_3^{-1} \xi_1 + \xi_3) x_\ell + (\mathcal{A}_4^{-1} \xi_2 + \xi_4) x_r} \rangle =$$

$$\langle (x_{1'}^{n-1} /\!\!/ \nabla^{1'}_{12})(x_3 + e^{-\epsilon a_3} x_r) e^{(\alpha_1+\alpha_3) a_\ell + \alpha_2 a_r + (\mathcal{A}_3^{-1} \xi_1 + \xi_3) x_\ell + \xi_2 x_r} \rangle =$$

$$\langle (x_{1'}^{n-1} /\!\!/ \nabla^{1'}_{12})(x_\ell + e^{-\epsilon a_3} x_r) e^{(\alpha_1+\alpha_3) a_\ell + \alpha_2 a_r + \mathcal{A}_3^{-1} \xi_1 x_\ell + \xi_2 x_r} \rangle =$$

$$\langle (x_{1'}^{n-1} /\!\!/ \nabla^{1'}_{1r})(x_\ell + e^{-\epsilon a_3} x_r) e^{(\alpha_1+\alpha_3) a_\ell + \mathcal{A}_3^{-1} \xi_1 x_\ell} \rangle =$$

$$(x_i^{n-1} /\!\!/ \nabla^i_{\ell r}) x_\ell + e^{-\epsilon a_\ell} \langle (x_i^{n-1} /\!\!/ \nabla^i_{1r}) e^{\alpha_1 + q^{-1} \xi_1 x_\ell} \rangle x_r$$

By induction (CHECK THIS AGAIN!) we find

$$x_i^n /\!\!/ \nabla^i_{\ell r} = \sum_{k+l=n} \frac{[k+l]!}{[k]![l]!} e^{-\epsilon a_\ell l} x_\ell^k x_r^l$$

Finally since $a_i^m x_i^n = a_u^m x_v^n /\!\!/ (m_{\mathbb{A}})^{uv}_i$ we have

$$a_i^m x_i^n /\!\!/ \nabla^i_{\ell r} = a_u^m x_v^n /\!\!/ \nabla^u_{12} \nabla^v_{34} /\!\!/ (m_{\mathbb{A}})^{13}_\ell (m_{\mathbb{A}})^{24}_r =$$

$$(a_1+a_2)^m\sum_{k+l=n}\frac{[k+l]!}{[k]![l]!}e^{-\epsilon a_3 l}x_3^k x_4^l /\!\!/ (m_{\mathbb{A}})^{13}_\ell (m_{\mathbb{A}})^{24}_r = (a_\ell+a_r)^m\sum_{k+l=n}\frac{[k+l]!}{[k]![l]!}e^{-\epsilon a_\ell l}x_\ell^k x_r^l$$

Therefore according to the previous computations:

$$\nabla^i_{\ell r} = \sum_{m,n}\frac{(\alpha_i a_i)^m(\xi_i x_i)^n}{m!n!} /\!\!/ \nabla^i_{\ell r} = \sum_{m,n}((a_\ell+a_r)\alpha_i)^m\sum_{k+l=n}\frac{[k+l]!}{[k]![l]!m!n!}e^{-\epsilon a_\ell l}x_\ell^k x_r^l \xi_i^{k+l} = (\Delta_{\mathbb{A}})^i_{\ell r}$$

It follows that $\Delta_{\mathbb{A}}$ satisfies the algebra morphism property. (Alternatively reformulate/relate to quantization of the 2d Lie algebra) □

The other properties of a purple category are easily checked explicitly. We have thus shown that the category of two step Gaussians is a purple category.

Furthermore, we notice that † does not actually occur when working with the usual morphisms excepting $P, \bar{P}$:

Lemma 8. *† is not in the subcategory $\tilde{\mathcal{G}}$ of $\mathcal{G}$ generated by unions of the morphisms D, J, μ, $\Delta_{\mathbb{A}}$, $\Delta_{\mathbb{B}}$, $m_{\mathbb{A}}$, $m_{\mathbb{B}}$, $1_{\mathbb{A}}$, $1_{\mathbb{B}}$, $S_{\mathbb{A}}$, $\bar{S}_{\mathbb{A}}$, $S_{\mathbb{B}}$, $\bar{S}_{\mathbb{B}}$, X, $\bar{X}$, $\epsilon_{\mathbb{A}}$, $\epsilon_{\mathbb{B}}$.*

Proof. At the first stage of contraction we do not get † since at $\epsilon = 0$ with $x = y = \xi = \eta = 0$ we get a copy of the one step Gaussian purple category. In that category we proved a lemma that † does not appear. Likewise at the second stage of contraction setting $\epsilon = 0$ and $a = b = \alpha = \beta = 0$ all formulas specialize again to the one step Gaussian purple category from the previous subsection. Therefore no † can occur in this stage either. □

This lemma defines the subcategory $\tilde{\mathcal{G}}$ mentioned in the main result. The functor $Z : \mathcal{R}_0 \to \mathcal{G}$ takes its values in $\tilde{\mathcal{G}}$.

Finally we aim to extend our functor to all of $\mathcal{R}$ by considering a square root of the element w. Although w can be computed explicitly we prefer computing it to 0-th order and arguing inductively that a unique square root exists.

We compute that

$$w_i = (X_{12}X_{34}) /\!\!/ S_1 /\!\!/ \bar{S}_3 /\!\!/ \bar{S}_3 /\!\!/ m_i^{1243} = B_i^{-1} + \mathcal{O}(\epsilon)$$

So we set $\bar{c}_i = B_i^{-\frac{1}{2}}(1+\sum_{k>0}P_k\epsilon^k)$ and argue that the P_k are uniquely determined by the requirement $\bar{c}_1\bar{c}_2 /\!\!/ m_i^{12} = w_i$. Indeed if we already know all P_ℓ for $\ell < k$ then we find

$$(B^{-\frac{1}{2}}(1+\sum_{\ell<k}P_\ell\epsilon^\ell + P_k\epsilon^k))_1(B^{-\frac{1}{2}}(1+\sum_{r<k}P_r\epsilon^r + P_k\epsilon^k))_2 /\!\!/ m_i^{12} =$$

$$(B^{-\frac{1}{2}}(1+\sum_{\ell<k}P_\ell\epsilon^\ell))_1(B^{-\frac{1}{2}}(1+\sum_{r<k}P_r\epsilon^r))_2 /\!\!/ m_i^{12} + 2B_i^{-1}(P_k)_i\epsilon^k + \mathcal{O}(\epsilon^{k+1}) = w_i$$

Hence, taking the coefficient of ϵ^k on both sides tells us what is P_k in terms of w_i and the previous P_ℓ, with $\ell < k$. The other desired equations for $\bar{c}_i$ all follow from the fact that w_i satisfies similar equations and has a unique square root. The inverse $c_i = B^{\frac{1}{2}} + \mathcal{O}(\epsilon)$ is shown to exist and be unique by a similar argument.

4 Proofs and Extras

4.1 More on rec-tangles

Lemma 9. *Any rec-tangle in $\mathcal{R}_0$ is equivalent to a composition of a union of fundamental rec-tangles $X^{\pm}, \Delta, S^{\pm}, 1, m, \epsilon$ as shown in figure 2*

Proof. First we push all vertices, u-turns and rotation out of the rectangles so that each rectangle intersects T in disjoint properly embedded intervals, each with rotation number 0. Pushing out things from a rectangle yields an equivalent one since it may be interpreted as contraction with the identity. Such rectangles may be reconstructed using Δ and S. Now cut away all crossings and reglue X's Since the total rotation number along each strand is 0 we may realize the rotation numbers on the edges by applying S repeatedly. Parts of Σ not containing R or T are irrelevant and can be added or removed as required. □

More importantly, we have a Reidemeister type theorem telling us when two rec-tangles presented this way are equivalent. This is important because using a functor to transfer the topological category of rectangles to a more algebraic category these will be the equations that need solving.

Theorem 5. *If two rec-tangles in $\mathcal{R}_0$ presented as a composition of basic rec-tangles are equivalent then their presentations must be related by a sequence of quasi-triangular category relations as shown graphically in Figure* **??**. *Conversely, all relations in a quasi-triangular category are valid in the category $\mathcal{R}$.*

Proof. Using (co)-associativity, the algebra morphism propery of Δ, the unit axioms, multiplicativity of S and the (co)-unit axioms we can assume both descriptions to be of the following form. First a couple of co-units, a couple of multiple Δ, a few X's all connected by integer powers of S to multiple m's and 1's. The co-units and the units can be assumed not to be composed with other things. Therefore the diffeomorphism must yield a bijection between the counits of the one description (they appear as the empty rectangles) and the counits of the other description. The same holds for the units of both descriptions. There must also be a bijection between the remaining rectangles.

Our next task is to show that using the quasi-triangular relations, pre- and post-contraction is well-defined and produces equivalent results. The other three operations generating equivalence of rec-tangles are clear. □

Finally extend these structure theorems to all rec-tangles, by adding $C, \bar{C}$.

4.2 Proof of the zipping theorem

Lemma 10.

$$\langle e^{c+\lambda z+z\ell+\zeta Qz}\rangle = \det(\tilde{Q})e^{c+\lambda\tilde{Q}\ell}$$

whenever the inverse matrix $\tilde{Q} = (1-Q)^{-1}$ *exists.*

Proof. Without loss of generality we assume $c = 0$. The proof is to simply expand both sides explicitly and describe the resulting terms combinatorially in terms of certain labelled graphs reminiscent of Feynman diagrams.

Using the geometric series and

$$\det\tilde{Q} = \det e^{\log(1-Q)^{-1}} = e^{tr\log(1-Q)^{-1}} = e^{\sum_k tr\frac{Q^k}{k}}$$

the right hand side may be written as

$$e^{\sum_k \lambda Q^k\ell + tr\frac{Q^k}{k}} = e^{\sum_k\sum_{i_1\dots i_k}\lambda_{i_0}Q_{0_1 i_1}Q_{i_1 i_2}\dots Q_{i_{k-1}i_k}\ell_{i_k}+\frac{1}{k}Q_{i_0 i_1}Q_{i_1 i_2}\dots Q_{i_k i_0}}$$

The left hand series can also be expanded:

$$\sum_{a_{ij},b_i,c_j=0}^{\infty}\left(\prod_{r,s\in I}\partial_{z_r}^{a_{rs}+b_r}\right)\prod_{i,j\in I}\frac{\lambda^{c_j}Q_{ij}^{a_{ij}}\ell^{b_i}}{a_{ij}!b_i!c_j!}z_j^{a_{ij}+c_j}|_{z=0}$$

Non-zero contributions are determined by a choice of the numbers a_{ij}, b_i, c_i for all $i,j\in I$ and a choice of matching each copy of ∂_r with a factor z_r for any $r\in I$. Such a choice contributes precisely $\frac{\lambda^{c_j}Q_{ij}^{a_{ij}}\ell^{b_i}}{a_{ij}!b_i!c_j!}$.

To describe both right and left hand sides more carefully we use simple Feynman type graphs that we call diagrams. A diagram is a directed graph D with only 1 and 2-valent edges and with vertices carrying labels $i\in I$. Each edge also carries a weight, for edge from i to j the weight is Q_{ij} if the vertices are both 2-valent and if i is an end-point the weight is λ_i and if j is, the weight is ℓ_j. In the latter two cases we require $i = j$. The weight $wt(D)$ of a diagram D is the product of the weights of all its edges. $\mathcal{D}$ is the set of all diagrams.

If $\mathcal{C}$ denotes the set of connected diagrams we may summarize the computation of the right hand side as:

$$RHS = e^{\sum_{G\in\mathcal{C}}\frac{wt(G)}{|Aut(G)|}}$$

$Aut(G)$ denotes the number of automorphisms of the diagram (with labelled vertices), it is of size k for a wheel graph with k edges and 1 for a path.

Next if we define an (edge) enumeration of a diagram D to be a choice of ordering the edges with a given weight. The number of edge enumerations of diagram D diagrams is called $\mathcal{E}(D)$. We claim that if $N_w(D)$ is the number of edges of weight w we have

$$LHS = \sum_{D\in\mathcal{D}} wt(D)\frac{\mathcal{E}(D)}{\prod_w N_w(D)!}$$

To explicitly see how the contributing terms are in bijection with the edge enumerated diagrams, for each i,j we arbitrarily match up a_{ij} pairs of a ∂_{z_i} with a z_j and identify each pair with an edge with beginning labeled j and end labelled i. Now matching up the edges according to how the term matches up the ∂_{z_r} with the z_r we get a diagram and after choosing arbitrary starting points in all the wheel components and an arbitrary order of the components we may enumerate each edge type according to its order of occurrence. This yields an enumerated diagram for the contributing term. Conversely, using the same conventions an enumerated diagram also produces a contributing term as we can read off precisely which ∂_r is paired with which z_r.

To finish our proof all that remains is to compute $\mathcal{D}$ in terms of the connected diagrams it consists of. We claim that

$$\mathcal{E}(D) = \frac{\prod_w N_w(D)!}{\prod_{G\in\mathcal{C}}|Aut(G)|^{n_G(D)}n_G(D)!}$$

where $n_G(D)$ is the number of copies of a connected diagram G occurring in D. This is because after ordering the edges of each weight type we overcounted by precisely reordering the connected components and also the automorphisms of each component.

Finally since $wt(D) = \prod_{G\in\mathcal{C}} wt(G)^{n_G(D)}$ we get

$$LHS = \sum_{D\in\mathcal{D}} \prod_{G\in\mathcal{C}} \frac{\left(\frac{wt(G)}{|Aut(G)|}\right)^{n_G(D)}}{n_G(D)!} = \sum_{(n_G)_{G\in\mathcal{C}}} \frac{\prod_{G\in\mathcal{C}} \left(\frac{wt(G)}{|Aut(G)|}\right)^{n_G}}{n_G!} = e^{\sum_{G\in\mathcal{C}} \frac{wt(G)}{|Aut(G)|}} = RHS$$

□

Theorem 6. *Suppose P is a polynomial in ζ and $I = J$ in the above. Assume $\tilde{Q} = (1-Q)^{-1}$ exists.*

$$\langle P(z,\zeta)e^{c+\lambda z+\zeta\ell+\zeta Qz}\rangle = \det(\tilde{Q})e^{c+\lambda\tilde{Q}\ell}\langle P(\tilde{Q}(z+\ell),\zeta+\lambda\tilde{Q})\rangle$$

Proof. To derive the theorem from Lemma 10 above we introduce auxiliary variables m, μ and write

$$\langle P(z,\zeta)e^{c+\lambda z+\zeta\ell+\zeta Qz}\rangle = P(\partial_\mu,\partial_m)e^{c+(\lambda+\mu)z+\zeta(\ell+m)+\zeta Qz}|_{m=\mu=0}$$

Since these differentiations commute with contraction, replacing ℓ by $\ell+m$ and λ by $\lambda+\mu$ the lemma says

$$\langle P(z,\zeta)e^{c+\lambda z+\zeta\ell+\zeta Qz}\rangle = \det(\tilde{Q})P(\partial_\mu,\partial_m)e^{c+(\lambda+\mu)\tilde{Q}(\ell+m)}|_{m=\mu=0} =$$

$$\det(\tilde{Q})P(\tilde{Q}(\ell+m),\partial_m)e^{c+\lambda\tilde{Q}(\ell+m)}|_{m=0} = \det(\tilde{Q})e^{c+\lambda\tilde{Q}\ell}\langle P(\tilde{Q}(\ell+z),\zeta+\lambda\tilde{Q})\rangle_{z,\zeta}$$

□

4.3 Connection to Hopf algebras and $U_h\mathfrak{sl}_2$

Define a Hopf algebra using ordering and PBW basis starting from the Gaussian tensors. Contract m with all the generators explicitly. Then identify this with $U_h\mathfrak{sl}_2$.

In this section we connect our invariant to the quantum group $U = U_h\mathfrak{sl}_2$ following the exposition by Chari-Pressley.

Recall that $U_h\mathfrak{sl}_2$ is generated over $\mathcal{C}[[h]]$ by $X^\pm$ and H subject to the relations $[H, X^\pm] = \pm 2X^\pm$ and $[X^+, X^-] = \frac{e^{hH}-e^{-hH}}{e^h-e^{-h}}$. We have the algebra $\mathcal{D}$ in axby order with relations $[x,a] = -x$ and $[y,b] = \epsilon y$ and $[b,x] = -\epsilon x$ and $[y,a] = y$, $[b,a] = 0$ and $[y,x] = B - A^{-1}$ where $A = e^{-\epsilon a}, B = e^{-b}$.

We rewrite the relations using the central element $t = \epsilon a - b$ and set $T = e^t$ as follows. $[y,x] = TA - A^{-1}$. When ϵ is invertible we get an algebra map $\phi : U_h\mathfrak{sl}_2 \to \mathcal{D}$ if we set $\phi(X^+) = (e^h - e^{-h})^{-1}T^{-1/2}y$ and $\phi(X^-) = x$ and $\phi(H) = h^{-1}(t/2 - \epsilon a)$ and as long as $\epsilon = 2h$:

$$\phi([H,X^+]) = \phi(2X^+) = 2Y = \epsilon Y/h = [h^{-1}(t/2-\epsilon a), Y] = [\phi(H),\phi(X^+)]$$

$$\phi([H,X^-]) = \phi(-2X^-) = -2x = -\epsilon x/h = [h^{-1}(t/2-\epsilon a), x] = [\phi(H),\phi(X^-)]$$

$$\phi([X^+,X^-]) = \phi(\frac{e^{hH}-e^{-hH}}{e^h-e^{-h}}) = \frac{e^{t/2-\epsilon a}-e^{-(t/2-\epsilon a)}}{e^h-e^{-h}} = [(e^h-e^{-h})^{-1}T^{-1/2}y, x] = [\phi(X^+),\phi(X^-)]$$

In the other direction we may write $\psi : \mathcal{D} \to U_h\mathfrak{sl}_2$ with $\psi(t) = \tau \in \mathcal{C}[[h]]$ and $\psi(a) = \epsilon^{-1}(\tau/2 - hH) = \frac{\tau}{2\epsilon} - \frac{H}{2}$ and $\psi(y) = (e^h - e^{-h})e^{\tau/2}X^+$ and $\psi(x) = X^-$.

Maybe the case $\tau = 0$ is sufficient to follow Chari-Pressley in saying that we obtain the R-matrix of $U_h\mathfrak{sl}_2$ and hence that universal quantum invariant by specializing ours.

4.4 Seifert formula for Alexander

Define the rec-tangle Bandersnatch as follows:

$$\text{Bandersnatch}^{ij}_k = C_3C_4\Delta^i_{\ell_1r_1}\Delta^j_{\ell_2r_2}/\!\!/\bar{S}_{r_1}/\!\!/S_{r_2}/\!\!/m^{\ell_1r_234r_1\ell_2}_k$$

Any Seifert surface for a knot K in band form may be presented using a tangle $\mathcal{B}$ with $2g$ components as follows.

$$K = \mathcal{B}/\!\!/^g_{j=1}\text{Bandersnatch}^{2j-1,2j}_j/\!\!/m^{12\ldots g}_1$$

We will show how at 0-th order in ϵ the invariant $Z(K)$ yields the Alexander polynomial using Seifert formula.

Recall (Burde-Zieschang p.107) that to construct the Alexander polynomial from a band presentation of a Seifert surface we start with a 2g-tangle $\mathcal{B}$, labelled 1..2g, encoding the bands. Suppose p_{ij} denotes the number of positive crossings where i-passes over j and $n_{i,j}$ the number of negative such. Then the Seifert matrix is $V_{ij} = p_{ij} - n_{ij}$.

We have at $\epsilon = 0$

$$Z(\text{Bandersnatch}^{ij}_k) = B_k^{-1}e^{\frac{(1-\mathcal{A}_j)\eta_i-(1-\mathcal{A}_i)\eta_j}{B_k\mathcal{A}_i}y_k+(\mathcal{A}_i(-1+\mathcal{A}_j)\mathcal{A}_j^{-1}\xi_i+(1-\mathcal{A}_i)\xi_j)x_k+\frac{(1-B_k-\mathcal{A}_j+B_k\mathcal{A}_j)(-1+B_k)\eta_j}{\mathcal{A}_jB_k}\xi_i+\frac{(-1+B_k+(\mathcal{A}_i+\mathcal{A}_j)(1-B_k))\eta}{\mathcal{A}_iB}}$$

Since this does not include terms α, β, a, b in the exponent, it follows that at $\epsilon = 0$ $Z(K) = Z(\mathcal{B}' /\!\!/_{j=1}^{g} \mathrm{BS}_j^{2j-1,2j} /\!\!/_{i=1}^{g} \mathrm{id}_1^i)$ where $\mathcal{B}' = X_{ij}^{p_{ij}} \bar{X}_{ij}^{n_{ij}}$ and $\mathrm{BS}_k^{ij} = B_k^{-1} e^{(1-B_k^{-1})(\xi_i \eta_j - \xi_j \eta_i)}$ is the specialization of $Z(\mathrm{Bandersnatch}_k^{ij})$ to the case where $\mathcal{A} = 1$ for every subscript. This is valid because at the first stage where we γ variables for K we see that the m_k^{ij} simply become renaming operations. Because the perturbation terms never include a or β. The Gaussian exponent vanishes alltogether so that the final m's in the formula are simply renamings, naming all components 1.

Our task is to relate the determinant coming from the second stage of zipping to the Seifert determinant formula: $\det(V^T - B_1 V)$ for the Alexander polynomial.

Recall that in order for V to be a Seifert matrix it should satisfy $V - V^T = F$ where F is the intersection form $F = \sum_{i=1}^{g} E_{2i-1,2i} - E_{2i,2i-1}$.

To set up the second stage of zipping for $Z(K)$ we arrange the variables into a $2g \times 2g$ block encoding the crossings. This block equals the Seifert matrix V since after the first stage the B^{-1} terms for the negative crossings have been set to 1. The other block in the block-diagonal 4g by 4g matrix is given by the BS and it is equal to $(1 - B_k^{-1})F$. So we need to show that

$$\det \begin{pmatrix} -V & I \\ I & -(B_k^{-1} - 1)F \end{pmatrix} = \det(V - B_k^{-1} V^T)$$

For block matrices with equal sized square blocks A, B, C, D satisfying $CD = DC$ we have that $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$. In our case $BC = I$ so the left hand side is

$$\det(V(B_k^{-1} - 1)F - I) = \det(-V(B_k^{-1} - 1) - F) = \det(-V(B_k^{-1} - 1) - V + V^T) = \det(-V B_k^{-1} + V^T)$$

here we used $F^2 = -I$ and the Seifert property of V.

4.5 Computation in polynomial time

Swap to central variable t and work with rational functions in t coefficients. Only algebra structure survives. Give computational upper bound in terms of zipping theorem. Polynomial time computation for tangles too? fix t-variables, one for every component.

4.6 Connection to previous invariant in polypoly paper

Took a quotient where central element Casimir = 0. Slightly different notion of snarls.