The 1D Zip Algebra

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$$(x\partial_y)^n(y\partial_z)^m\Big|_{y=0} = \delta_{nm}\, n!\,(x\partial_z)^n$$

$$e^{x\partial_y}(x\partial_y)^n\, e^{y\partial_z}(y\partial_z)^m\Big|_{y=0} =$$

$$=(x\partial_y)^n e^{x\partial_y} e^{y\partial_z}(y\partial_z)^m\Big|_{y=0}$$

$$=(x\partial_y)^n\, e^{(x+y)\partial_z}((x+y)\partial_z)^m\Big|_{y=0}$$

$$=\sum_{k=0}^{n}\binom{n}{k} x^n e^{x\partial_z} x^{m-k}\partial_z \ldots$$

$$e^{(1+\alpha)x\partial_y}\, e^{(1+\beta)y\partial_z}\Big|_{y=0} =$$

$$e^{(1+\beta)(y+(1+\alpha)x)\partial_z}\Big|_{y=0} = e^{(1+\alpha)(1+\beta)x\partial_z}$$

$$= e^{x\partial_z}\, e^{\alpha x\partial_z}\, e^{\beta x\partial_z}\, e^{\alpha\beta x\partial_z}$$

So w/ $s = x\partial_y$,

$$s^n \cdot s^m = \sum_{k=0}^{\min(n,m)} \underline{\qquad}\; s^{n+m-k}$$

From ZipAlgebra.nb:

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In[15]:= Table[Last@Zip_{η}[𝔼[ξ y + η z, (ξ y)^m (η z)^n]] /. ξ → 1,  {m, 0, 4}, {n, 0, 4}]  //  MatrixForm
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Out[15]//MatrixForm=

$$\begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ z & z(1+z) & z^2(2+z) & z^3(3+z) & z^4(4+z) \\ z^2 & z^2(2+z) & z^2(2+4z+z^2) & z^3(6+6z+z^2) & z^4(12+8z+z^2) \\ z^3 & z^3(3+z) & z^3(6+6z+z^2) & z^3(6+18z+9z^2+z^3) & z^4(24+36z+12z^2+z^3) \\ z^4 & z^4(4+z) & z^4(12+8z+z^2) & z^4(24+36z+12z^2+z^3) & z^4(24+96z+72z^2+16z^3+z^4) \end{pmatrix}$$

In[35]:= $\mathbf{B}\left[\begin{smallmatrix} n\_ \\ k\_ \end{smallmatrix}\right]$:= Binomial[n, k];

f[m_, n_] := Simplify$\left[\sum_{k=0}^{\mathrm{Min}[m,n]} k!\, \mathbf{B}\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] \mathbf{B}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] z^{m+n-k}\right]$;

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Table[f[m, n], {m, 0, 4}, {n, 0, 4}] // MatrixForm
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Out[37]//MatrixForm=

$$\begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ z & z(1+z) & z^2(2+z) & z^3(3+z) & z^4(4+z) \\ z^2 & z^2(2+z) & z^2(2+4z+z^2) & z^3(6+6z+z^2) & z^4(12+8z+z^2) \\ z^3 & z^3(3+z) & z^3(6+6z+z^2) & z^3(6+18z+9z^2+z^3) & z^4(24+36z+12z^2+z^3) \\ z^4 & z^4(4+z) & z^4(12+8z+z^2) & z^4(24+36z+12z^2+z^3) & z^4(24+96z+72z^2+16z^3+z^4) \end{pmatrix}$$