

Etingof on Tannakian Interpretation of Elliptic
Infinitesimal Braid Lie Algebras

(With B. Enriquez)

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$T = S^1 \times S^1$ $C_n(T) =$ conf of n distinct labeled points in T "I'm cultured"
 n basept. $= T^n \setminus (\text{Diags})$
 $\pi_1(C_n(T), x) =: PB_{1,n}^x =$ The pure elliptic braid group

$\widehat{PB}_{1,n}^x$: The pro-unipotent completion over \mathbb{C}
 $=$ group-like elements of $\varprojlim_{N \rightarrow \infty} EG / (\ker E)^N$

$\text{Lie } \widehat{PB}_{1,n}^x$ — a pro-nilpotent Lie algebra,
 has LCS filtration, has gr:

$\text{gr Lie } \widehat{PB}_{1,n}^x =$ computed by Bezkovnikov.

Def $t_{1,n}$ is the Lie alg/ \mathbb{C} w/ generators

x_i, y_i, t_{ij} modulo

$$[x_i, x_j] = [y_i, y_j] = 0$$

$$t_{ij} = [x_i, y_j]$$

i, j, k
 distinct

$$[x_i, y_i] = - \sum_{j \neq i} t_{ij}$$

$$[x_k, t_{ij}] = [y_k, t_{ij}] = 0$$

$$\Rightarrow [x_i + x_j, t_{ij}] = 0 = [y_i + y_j, t_{ij}]$$

Thm (Bezrukavnikov ~ 1995)

1. $\text{gr Lie } \hat{PB}_{1,n}^x \cong \mathfrak{t}_{1,n}$

2. $\text{Lie } \hat{PB}_{1,n}^x \cong \hat{\mathfrak{t}}_{1,n}$ as LCS-filtered Lie algebras, w/ isomorphism that lifts the previous one.

2005 Etingof Calaque Enriquez: A KZ -like proof.

- construct a universal Knizhnik-Zamolodchikov - Bernard

(KZB) connection.

Want $\hat{PB}_{1,n}^x = \exp(\hat{\mathfrak{t}}_{1,n})$
 \cup
 $PB_{1,n}^x \xrightarrow{\text{construct using a flat connection}}$

construct an $\exp(\hat{\mathfrak{t}}_{1,n})$ -bundle on $C_n(T)$

Take $T = E_\tau = \mathbb{C} / (\mathbb{Z} + \tau\mathbb{Z}) \quad \tau \in \mathbb{C}^+$

an elliptic curve.

Take trivial bundle on $(\mathbb{C}/\mathbb{Z})^n$

construct bundle on E_τ^n using clutching

maps $e^{-2\pi i x_j}$:

so sections are functions $\mathbb{C}^n \rightarrow \text{Exp}(\widehat{\mathfrak{t}}_{1,n})$

s.t. $f(\vec{z} + e_j) = f(\vec{z}), f(\vec{z} + \nu e_j) = e^{-2\pi i x_j} f(\vec{z})$

$\theta(z) = \theta(z, \tau)$ are θ -functions normalized by $\theta'(0) = 1$

$\nabla_{KZB} = d + w$ p_i : coord on i 'th copy of E_ν .

$$w = - \sum_{i \neq j} \left(\frac{\theta(p_i - p_j + \alpha(x_i)) \alpha(x_j)}{\theta(p_i - p_j) \theta(\alpha(x_j))} y_j \right) dp_i$$

Thm (CE-E)

1. ∇_{KZB} is flat

2. Monodromy defines ^{homo} morphism

$$PB_{1,n}^X \rightarrow \text{Exp}(\widehat{\mathfrak{t}}_{1,n})$$

which gives rise to

$$\widehat{PB}_{1,n}^X \cong \text{Exp}(\widehat{\mathfrak{t}}_{1,n})$$

KZ is obtained if

$$\tau = w_1/w_2, w_{1,2} \rightarrow \infty$$

?

Tannakian interpretation $X = C_n(E)$

$\text{VBFC}_{\text{unip}}(X)$ cat. of vect bdlcs on X
_{horo/algebraic}

with unip flat connection.

$LS_{unip}(X)$ - unip. local systems on X .

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