

The tensorial double at $\epsilon^2 = 0$

```
In[ ]:=  $\epsilon$  /:  $e^{d \cdot}$  /;  $d > 1$  := 0;
```

```
In[ ]:= CF[ $\mathcal{E}$ _] := ExpandDenominator@
  ExpandNumerator@Together[Expand[ $\mathcal{E}$ ] /.  $e^x e^y \rightarrow e^{x+y}$  /.  $e^x \rightarrow e^{CF[x]}$ ];
K $\delta$  /: K $\delta_{i,j}$  := If[i == j, 1, 0];
```

Zip and Bind

```
In[ ]:=  $\mathbb{E}$  /:  $\mathbb{E}[L1_, Q1_, P1_] \equiv \mathbb{E}[L2_, Q2_, P2_] :=$ 
  CF[L1 == L2]  $\wedge$  CF[Q1 == Q2]  $\wedge$  CF[Normal[P1 - P2] == 0];
 $\mathbb{E}$  /:  $\mathbb{E}[L1_, Q1_, P1_] \mathbb{E}[L2_, Q2_, P2_] := \mathbb{E}[L1 + L2, Q1 + Q2, P1 * P2];$ 
```

```
In[ ]:= {t*, b*, y*, a*, x*, z*} = { $\tau, \beta, \eta, \alpha, \xi, \zeta$ };
{ $\tau^*, \beta^*, \eta^*, \alpha^*, \xi^*, \zeta^*$ } = {t, b, y, a, x, z}; ( $u_{-i}$ )* := ( $u^*$ )i;
```

```
In[ ]:= Zip[ $\{$ ][P_] := P; Zip[ $\{\xi, \zeta\}$ ][P_] := (Expand[P /. Zip[ $\{\xi, \zeta\}$ ] /.  $f_{-} \cdot \xi^{d \cdot} \rightarrow \partial_{\{\xi^*, d\}} f$ ] /.  $\zeta^* \rightarrow 0$ )
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard the L variables as scalars.

```
In[ ]:= QZip[ $\xi\zeta$ _List, simp_]@ $\mathbb{E}[L_, Q_, P_] := Module[{ $\xi, z, zs, c, ys, \eta s, qt, zrule, Q1, Q2$ },
  zs = Table[ $\xi^*$ , { $\xi, \zeta$ }]];
  c = Q /. Alternatives@@( $\xi\zeta \cup zs$ )  $\rightarrow 0$ ;
  ys = Table[ $\partial_{\xi}(Q /. Alternatives@@zs \rightarrow 0)$ , { $\xi, \zeta$ }]];
   $\eta s$  = Table[ $\partial_z(Q /. Alternatives@@\xi\zeta \rightarrow 0)$ , {z, zs}];
  qt = Inverse@Table[K $\delta_{z, \xi^*} - \partial_{z, \xi} Q$ , { $\xi, \zeta$ }, {z, zs}];
  zrule = Thread[zs  $\rightarrow$  qt.(zs + ys)];
  Q2 = (Q1 = c +  $\eta s$ .zs /. zrule) /. Alternatives@@zs  $\rightarrow 0$ ;
  simp /@  $\mathbb{E}[L, Q2, Det[qt] e^{-Q2} Zip_{\xi\zeta}[e^{Q1}(P /. zrule)]]$ ];
QZip[ $\xi\zeta$ _List] := QZip[ $\xi\zeta$ , CF];$ 
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single “ P ”. Here the z ’s are b and α and the ζ ’s are β and a .

```

In[ ]:= LZip $\xi_S$ _List,simp_@E[L_, Q_, P_] :=
Module[{ $\xi$ , z, zs, c, ys,  $\eta_S$ ,  $\eta_S$ rule, lt, zrule, L1, L2, Q1, Q2},
(*Print["LZipping"];*)
(* $\xi_S$ //Echo;*)
zs = Table[ $\xi^*$ , { $\xi$ ,  $\xi_S$ };]
c = L /. Alternatives@@( $\xi_S \cup zs$ )  $\rightarrow$  0;
ys = Table[ $\partial_\xi$ (L /. Alternatives@@zs  $\rightarrow$  0), { $\xi$ ,  $\xi_S$ };]
 $\eta_S$  = Table[ $\partial_z$ (L /. Alternatives@@ $\xi_S \rightarrow$  0), {z, zs}];]
 $\eta_S$ rule = Table[z*  $\rightarrow$   $\partial_z$ (L /. Alternatives@@ $\xi_S \rightarrow$  0), {z, zs}]; (*NEW*)
lt = Inverse@Table[K $\delta_{z,\xi^*} - \partial_{z,\xi}L$ , { $\xi$ ,  $\xi_S$ }, {z, zs}];]
zrule = Thread[zs  $\rightarrow$  lt.(zs + ys)];]
L1 = c +  $\eta_S$ .zs /. zrule;]
L2 = L1 /. Alternatives@@zs  $\rightarrow$  0;]
Q1 = Q /. Join[zrule,  $\eta_S$ rule];]
Q2 = Q1 /. Join[{Alternatives@@zs  $\rightarrow$  0},  $\eta_S$ rule];]
(*{Det[lt]e-L2-Q2,Zip $\xi_S$ [eL1+Q1(P/.zrule)]}//Echo;*)
simp /@ E[L2, Q2, Det[lt] e-L2-Q2 Zip $\xi_S$ [eL1+Q1(P/.zrule)]]];]
LZip $\xi_S$ _List := LZip $\xi_S$ ,CF;

```

```

In[ ]:= Bind_{ }[L_, R_] := L R;
Bind_{is_}[L_ $\mathbb{E}$ , R_ $\mathbb{E}$ ] := Module[{n},
Times[
L /. Table[(v : t | b | a | x | y)i  $\rightarrow$  vni, {i, {is}}],
R /. Table[(v :  $\tau$  |  $\beta$  |  $\alpha$  |  $\xi$  |  $\eta$ )i  $\rightarrow$  vni, {i, {is}}]
] // LZipFlatten@Table[{ $\tau_{ni$ ,  $\beta_{ni$ ,  $\alpha_{ni$ }, {i, {is}}] // QZipFlatten@Table[{ $\xi_{ni$ , yni}, {i, {is}}] ];]
B_L_List := Bind_{ }; B_is_ := Bind_{is};

```

The two halves

```

In[ ]:= (*Hopf algebra on the a,x side*)
tami,j $\rightarrow$ k := E[( $\alpha_i + \alpha_j$ ) ak, (e- $\alpha_j$   $\xi_i + \xi_j$ ) xk, 1]
ta $\Delta$ i $\rightarrow$ j,k := E[ $\alpha_i$  (aj + ak),  $\xi_i$  (xj + xk), 1 + e  $\xi_i$  xk (-aj +  $\frac{1}{2}$   $\xi_i$  xj)]
taSi := E[- $\alpha_i$  ai, -e $\alpha_i$   $\xi_i$  xi, 1 - e e $\alpha_i$   $\xi_i$  xi (ai +  $\frac{1}{2}$  e $\alpha_i$   $\xi_i$  xi)]
taSii := E[- $\alpha_i$  ai, -e $\alpha_i$   $\xi_i$  xi, 1 - e e $\alpha_i$   $\xi_i$  xi (ai - 1 +  $\frac{1}{2}$  e $\alpha_i$   $\xi_i$  xi)]

(*Hopf algebra on the y,b side*)
tbmi,j $\rightarrow$ k := E[( $\beta_i + \beta_j$ ) bk, ( $\eta_i + \eta_j$ ) yk, 1 - e  $\eta_j$  yk  $\beta_i$ ]
tb $\Delta$ i $\rightarrow$ j,k := E[ $\beta_i$  (bj + bk),  $\eta_i$  (e-bk yj + yk), 1 +  $\frac{1}{2}$  e  $\eta_i^2$  yj yk e-bk]
tbSi := E[- $\beta_i$  bi, -ebi  $\eta_i$  yi, 1 - e ebi  $\eta_i$  yi ( $\beta_i$  +  $\frac{1}{2}$  ebi  $\eta_i$  yi)]
tbSii := E[- $\beta_i$  bi, -ebi  $\eta_i$  yi, 1 - e ebi  $\eta_i$  yi ( $\beta_i$  - 1 +  $\frac{1}{2}$  ebi  $\eta_i$  yi)]

```

First check that on the generators this agrees with our conventions in SLPortfolio.pdf with $\hbar = \gamma = 1$

```
In[*]:= {
  "[a,x] = " (E[0, 0, a2 x1] ~ B1,2 ~ tam1,2->1) [[3]] - (E[0, 0, a1 x2] ~ B1,2 ~ tam1,2->1) [[3]],
  "[b,y] = " (E[0, 0, y2 b1] ~ B1,2 ~ tbm1,2->1) [[3]] - (E[0, 0, y1 b2] ~ B1,2 ~ tbm1,2->1) [[3]]
} /. {e -> E, z_1 -> z} // Simplify

{
  "\Delta(a) = " (E[0, 0, a1] ~ B1 ~ taDelta1->1,2) [[3]],
  "\Delta(x) = " (E[0, 0, x1] ~ B1 ~ taDelta1->1,2) [[3]],
  "\Delta(b) = " (E[0, 0, b1] ~ B1 ~ tbDelta1->1,2) [[3]],
  "\Delta(y) = " (E[0, 0, y1] ~ B1 ~ tbDelta1->1,2) [[3]]
} /. {e -> E} // Simplify

{
  "S(a) = " (E[0, 0, a1] ~ B1 ~ taS1) [[3]],
  "S(x) = " (E[0, 0, x1] ~ B1 ~ taS1) [[3]],
  "S(b) = " (E[0, 0, b1] ~ B1 ~ tbS1) [[3]],
  "S(y) = " (E[0, 0, y1] ~ B1 ~ tbS1) [[3]]
} /. {e -> E, z_1 -> z} // Simplify
```

```
Out[*]:= {- [a,x] = x, - [b,y] = y \in}
```

```
Out[*]:= { \Delta(a) = (a1 + a2), \Delta(x) = (x1 + (1 - \epsilon a1) x2), \Delta(b) = (b1 + b2), \Delta(y) = (e^{-b2} y1 + y2) }
```

```
Out[*]:= {- S(a) = a, - S(x) = (x + a x \epsilon), - S(b) = b, - S(y) = e^b y }
```

Hopf algebra axioms on both sides separately

(co)-associativity on both sides

```
In[*]:= { (taDelta1->1,2 ~ B2 ~ taDelta2->2,3) \equiv (taDelta1->1,3 ~ B1 ~ taDelta1->1,2),
  (tbDelta1->1,2 ~ B2 ~ tbDelta2->2,3) \equiv (tbDelta1->1,3 ~ B1 ~ tbDelta1->1,2),
  (tam1,2->1 ~ B1 ~ tam1,3->1) \equiv (tam2,3->2 ~ B2 ~ tam1,2->1),
  (tbm1,2->1 ~ B1 ~ tbm1,3->1) \equiv (tbm2,3->2 ~ B2 ~ tbm1,2->1) }
```

```
Out[*]:= {True, True, True, True}
```

Δ is an algebra morphism

```
In[*]:= { tam1,2->1 ~ B1 ~ taDelta1->1,2 \equiv (taDelta1->1,3 taDelta2->2,4) ~ B1,2,3,4 ~ (tam3,4->2 tam1,2->1),
  tbm1,2->1 ~ B1 ~ tbDelta1->1,2 \equiv (tbDelta1->1,3 tbDelta2->2,4) ~ B1,2,3,4 ~ (tbm3,4->2 tbm1,2->1) } /. e -> E
```

```
Out[*]:= {True, True}
```

S is convolution inverse of id

```
In[*]:= { (taDelta1->1,2 ~ B1 ~ taS1) ~ B1,2 ~ tam1,2->1, (taDelta1->1,2 ~ B2 ~ taS2) ~ B1,2 ~ tam1,2->1 } /. e -> E
  { (tbDelta1->1,2 ~ B1 ~ tbS1) ~ B1,2 ~ tbm1,2->1, (tbDelta1->1,2 ~ B2 ~ tbS2) ~ B1,2 ~ tbm1,2->1 } /. e -> E
```

```
Out[*]:= {E[0, 0, 1], E[0, 0, 1]}
```

```
Out[*]:= {E[0, 0, 1], E[0, 0, 1]}
```

S is the inverse of S

```
In[ ]:= {taSi1~B1~taS1 ≡ E[a1 α1, x1 ξ1, 1], taS1~B1~taSi1 ≡ E[a1 α1, x1 ξ1, 1]} /. e → E
      {tbSi1~B1~tbS1 ≡ E[b1 β1, y1 η1, 1], tbS1~B1~tbSi1 ≡ E[b1 β1, y1 η1, 1]} /. e → E
```

```
Out[ ]:= {True, True}
```

```
Out[ ]:= {True, True}
```

S is an algebra anti-(co)morphism

```
In[ ]:= {tam1,2→1~B1~taS1 ≡ (taS1 taS2)~B1,2~tam2,1→1,
      tbm1,2→1~B1~tbS1 ≡ (tbS1 tbS2)~B1,2~tbm2,1→1} /. e → E
      {taS1~B1~taΔ1→1,2 ≡ taΔ1→2,1~B1,2~(taS1 taS2),
      tbS1~B1~tbΔ1→1,2 ≡ tbΔ1→2,1~B1,2~(tbS1 tbS2)} /. e → E
```

```
Out[ ]:= {True, True}
```

```
Out[ ]:= {True, True}
```

Pairing

```
In[ ]:= tP_{i,j}_ := E[β_i α_j, η_i ξ_j, 1 + 1/4 ε η_i^2 ξ_j^2]
```

```
In[ ]:= qfac[k_, q_] := (1 - q)^{-k} QPochhammer[q, q, k] // FunctionExpand
qfe[k_] := Normal[Series[qfac[k, E^ρ], {ρ, 0, 1}]] /. {ρ → ε}
Table[E[0, 0, y1^r b1^s a2^t x2^u]~B1,2~tP1,2 ≡ E[0, 0, Kδ_{r,u} Kδ_{s,t} qfe[r] s!],
      {r, 0, 4}, {s, 0, 4}, {t, 0, 4}, {u, 0, 4}] // Flatten // Union
```

```
Out[ ]:= {True}
```

Pairing axioms

```
In[ ]:= {(tbm1,2→1 E[α3 a3, ξ3 x3, 1])~B1,3~tP1,3 ≡
      (E[β1 b1, η1 y1, 1] E[β2 b2, η2 y2, 1] taΔ3→4,5)~B1,4~tP1,4~B2,5~tP2,5,
      (tbΔ1→1,2 E[α3 a3, ξ3 x3, 1] E[α4 a4, ξ4 x4, 1])~B1,3~tP1,3~B2,4~tP2,4 ≡
      (E[β1 b1, η1 y1, 1] tam3,4→3)~B1,3~tP1,3}
```

```
Out[ ]:= {True, True}
```

```
In[ ]:= {(tbS1 E[α2 a2, ξ2 x2, 1])~B1,2~tP1,2 ≡ (E[β1 b1, η1 y1, 1] taS2)~B1,2~tP1,2,
      (tbSi1 E[α2 a2, ξ2 x2, 1])~B1,2~tP1,2 ≡ (E[β1 b1, η1 y1, 1] taSi2)~B1,2~tP1,2}
```

```
Out[ ]:= {True, True}
```

The Double

The double multiplication (should really bind the a's and b's separately)

In[*]:= **Simplify /@ Expand /@**

$$\left(\mathbb{E} [\beta_i b_i + \alpha_j a_j, \eta_i y_i + \xi_j x_j, 1] \left(\mathbf{ta}_{\Delta_i \rightarrow h_1, h_2} \sim \mathbf{B}_{h_2} \sim \mathbf{ta}_{\Delta_{h_2 \rightarrow h_2, h_3}} \right) \left(\mathbf{tb}_{\Delta_j \rightarrow t_1, t_2} \sim \mathbf{B}_{t_2} \sim \mathbf{tb}_{\Delta_{t_2 \rightarrow t_2, t_3}} \right) \sim \right. \\ \left. \mathbf{B}_{h_3} \sim \mathbf{ta}_{\mathbf{Si}_{h_3}} \sim \mathbf{B}_{t_1, h_3} \sim \left(\mathbf{tP}_{t_1, h_3} \right) \sim \mathbf{B}_{t_3, h_1} \sim \left(\mathbf{tP}_{t_3, h_1} \right) \sim \right. \\ \left. \mathbf{B}_{h_2, j, i, t_2} \sim \left(\mathbf{tam}_{h_2, j \rightarrow k} \mathbf{tbm}_{i, t_2 \rightarrow k} \right) / \cdot \{ \mathbf{e} \rightarrow \mathbf{E}, \mathbf{u}_{-k} :> \mathbf{u}_k \} \right)$$

$$\text{Out[*]} = \mathbb{E} \left[\mathbf{a}_k (\alpha_i + \alpha_j) + \mathbf{b}_k (\beta_i + \beta_j), \mathbf{y}_k (\eta_i + e^{-\alpha_i} \eta_j) + e^{-b_k} (-1 + e^{b_k}) \eta_j \xi_i + \mathbf{x}_k (e^{-\alpha_j} \xi_i + \xi_j), \right. \\ \left. \frac{1}{4} e^{-2b_k - \alpha_i - \alpha_j} \left(2 e^{b_k} \in \mathbf{y}_k \eta_j (-2 e^{b_k + \alpha_j} \beta_i + (2 e^{b_k} \mathbf{x}_k + e^{\alpha_j} (-3 + e^{b_k})) \eta_j) \xi_i \right) + \right. \\ \left. e^{\alpha_i} \left(2 e^{b_k} \in \mathbf{x}_k \xi_i (-2 e^{b_k} \beta_j + (-3 + e^{b_k}) \eta_j) \xi_i \right) + \right. \\ \left. e^{\alpha_j} \left(4 e^{2b_k} + 4 e^{b_k} \in \mathbf{a}_k \eta_j \xi_i + (3 - 4 e^{b_k} + e^{2b_k}) \in \eta_j^2 \xi_i^2 \right) \right]$$

In[*]:= **(*Deriving tds using tdm*)**

$$\left(\left(\mathbf{tbSi}_1 \mathbf{taS}_2 \right) \sim \mathbf{B}_{1,2} \sim \mathbf{tdm}_{2,1 \rightarrow i} \right) / \cdot \{ \mathbf{e} \rightarrow \mathbf{E}, \mathbf{z}_{-1|2} \rightarrow \mathbf{z}_i \}$$

$$\text{Out[*]} = \mathbb{E} \left[-\mathbf{a}_i \alpha_i - \mathbf{b}_i \beta_i, e^{-b_i} \left(-e^{2b_i + \alpha_i} \mathbf{y}_i \eta_i - e^{b_i + \alpha_i} \mathbf{x}_i \xi_i - e^{b_i + \alpha_i} \eta_i \xi_i + e^{2b_i + \alpha_i} \eta_i \xi_i \right), \right. \\ \left. \frac{1}{4} e^{-2b_i} \left(4 e^{2b_i} + 4 e^{3b_i + \alpha_i} \in \mathbf{y}_i \eta_i - 4 e^{3b_i + \alpha_i} \in \mathbf{y}_i \beta_i \eta_i - 2 e^{4b_i + 2\alpha_i} \in \mathbf{y}_i^2 \eta_i^2 - \right. \right. \\ \left. \left. 4 e^{2b_i + \alpha_i} \in \mathbf{a}_i \mathbf{x}_i \xi_i - 4 e^{2b_i + \alpha_i} \in \mathbf{x}_i \beta_i \xi_i + 4 e^{2b_i + \alpha_i} \in \eta_i \xi_i - 4 e^{3b_i + \alpha_i} \in \eta_i \xi_i + \right. \right. \\ \left. \left. 4 e^{3b_i + \alpha_i} \in \mathbf{a}_i \eta_i \xi_i - 4 e^{3b_i + 2\alpha_i} \in \mathbf{x}_i \mathbf{y}_i \eta_i \xi_i - 4 e^{2b_i + \alpha_i} \in \beta_i \eta_i \xi_i + 4 e^{3b_i + \alpha_i} \in \beta_i \eta_i \xi_i - \right. \right. \\ \left. \left. 2 e^{3b_i + 2\alpha_i} \in \mathbf{y}_i \eta_i^2 \xi_i + 6 e^{4b_i + 2\alpha_i} \in \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{2b_i + 2\alpha_i} \in \mathbf{x}_i^2 \xi_i^2 - 2 e^{2b_i + 2\alpha_i} \in \mathbf{x}_i \eta_i \xi_i^2 + \right. \right. \\ \left. \left. 6 e^{3b_i + 2\alpha_i} \in \mathbf{x}_i \eta_i \xi_i^2 - e^{2b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 + 4 e^{3b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 - 3 e^{4b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 \right) \right]$$

In[*]:= **(*Deriving tdDelta using tdm*)**

$$\left(\mathbf{tb}_{\Delta_i \rightarrow 3, 1} \mathbf{ta}_{\Delta_i \rightarrow 2, 4} \right) \sim \mathbf{B}_{1,2,3,4} \sim \left(\mathbf{tdm}_{3,4 \rightarrow k} \mathbf{tdm}_{1,2 \rightarrow j} \right)$$

$$\text{Out[*]} = \mathbb{E} \left[\mathbf{a}_j \alpha_i + \mathbf{a}_k \alpha_i + \mathbf{b}_j \beta_i + \mathbf{b}_k \beta_i, e^{-b_j} \left(e^{b_j} \mathbf{y}_j \eta_i + \mathbf{y}_k \eta_i + e^{b_j} \mathbf{x}_j \xi_i + e^{b_j} \mathbf{x}_k \xi_i \right), \right. \\ \left. \frac{1}{2} e^{-b_j} \left(2 e^{b_j} + \in \mathbf{y}_j \mathbf{y}_k \eta_i^2 - 2 e^{b_j} \in \mathbf{a}_j \mathbf{x}_k \xi_i + e^{b_j} \in \mathbf{x}_j \mathbf{x}_k \xi_i^2 \right) \right]$$

$$\text{In[*]} := \mathbf{tdm}_{i,j \rightarrow k} := \mathbb{E} \left[\mathbf{a}_k (\alpha_i + \alpha_j) + \mathbf{b}_k (\beta_i + \beta_j), \mathbf{y}_k (\eta_i + e^{-\alpha_i} \eta_j) + e^{-b_k} (-1 + e^{b_k}) \eta_j \xi_i + \mathbf{x}_k (e^{-\alpha_j} \xi_i + \xi_j), \right. \\ \left. \frac{1}{4} e^{-2b_k - \alpha_i - \alpha_j} \left(2 e^{b_k} \in \mathbf{y}_k \eta_j (-2 e^{b_k + \alpha_j} \beta_i + (2 e^{b_k} \mathbf{x}_k + e^{\alpha_j} (-3 + e^{b_k})) \eta_j) \xi_i \right) + e^{\alpha_i} \left(2 e^{b_k} \in \mathbf{x}_k \xi_i \right. \right. \\ \left. \left. (-2 e^{b_k} \beta_j + (-3 + e^{b_k}) \eta_j) \xi_i \right) + e^{\alpha_j} \left(4 e^{2b_k} + 4 e^{b_k} \in \mathbf{a}_k \eta_j \xi_i + (3 - 4 e^{b_k} + e^{2b_k}) \in \eta_j^2 \xi_i^2 \right) \right] \\ \mathbf{td}_{\Delta_i \rightarrow j, k} := \mathbb{E} \left[\mathbf{a}_j \alpha_i + \mathbf{a}_k \alpha_i + \mathbf{b}_j \beta_i + \mathbf{b}_k \beta_i, e^{-b_j} \left(e^{b_j} \mathbf{y}_j \eta_i + \mathbf{y}_k \eta_i + e^{b_j} \mathbf{x}_j \xi_i + e^{b_j} \mathbf{x}_k \xi_i \right), \right. \\ \left. \frac{1}{2} e^{-b_j} \left(2 e^{b_j} + \in \mathbf{y}_j \mathbf{y}_k \eta_i^2 - 2 e^{b_j} \in \mathbf{a}_j \mathbf{x}_k \xi_i + e^{b_j} \in \mathbf{x}_j \mathbf{x}_k \xi_i^2 \right) \right] \\ \mathbf{tdS}_i := \mathbb{E} \left[-\mathbf{a}_i \alpha_i - \mathbf{b}_i \beta_i, e^{-b_i} \left(-e^{2b_i + \alpha_i} \mathbf{y}_i \eta_i - e^{b_i + \alpha_i} \mathbf{x}_i \xi_i - e^{b_i + \alpha_i} \eta_i \xi_i + e^{2b_i + \alpha_i} \eta_i \xi_i \right), \right. \\ \left. \frac{1}{4} e^{-2b_i} \left(4 e^{2b_i} + 4 e^{3b_i + \alpha_i} \in \mathbf{y}_i \eta_i - 4 e^{3b_i + \alpha_i} \in \mathbf{y}_i \beta_i \eta_i - 2 e^{4b_i + 2\alpha_i} \in \mathbf{y}_i^2 \eta_i^2 - \right. \right. \\ \left. \left. 4 e^{2b_i + \alpha_i} \in \mathbf{a}_i \mathbf{x}_i \xi_i - 4 e^{2b_i + \alpha_i} \in \mathbf{x}_i \beta_i \xi_i + 4 e^{2b_i + \alpha_i} \in \eta_i \xi_i - 4 e^{3b_i + \alpha_i} \in \eta_i \xi_i + \right. \right. \\ \left. \left. 4 e^{3b_i + \alpha_i} \in \mathbf{a}_i \eta_i \xi_i - 4 e^{3b_i + 2\alpha_i} \in \mathbf{x}_i \mathbf{y}_i \eta_i \xi_i - 4 e^{2b_i + \alpha_i} \in \beta_i \eta_i \xi_i + 4 e^{3b_i + \alpha_i} \in \beta_i \eta_i \xi_i - \right. \right. \\ \left. \left. 2 e^{3b_i + 2\alpha_i} \in \mathbf{y}_i \eta_i^2 \xi_i + 6 e^{4b_i + 2\alpha_i} \in \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{2b_i + 2\alpha_i} \in \mathbf{x}_i^2 \xi_i^2 - 2 e^{2b_i + 2\alpha_i} \in \mathbf{x}_i \eta_i \xi_i^2 + \right. \right. \\ \left. \left. 6 e^{3b_i + 2\alpha_i} \in \mathbf{x}_i \eta_i \xi_i^2 - e^{2b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 + 4 e^{3b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 - 3 e^{4b_i + 2\alpha_i} \in \eta_i^2 \xi_i^2 \right) \right]$$

First check the double formulas on the generators agree with SL2Portfolio.pdf:

```

In[*]:= {
  " [a,y] = " ((E[0, 0, y2 a1] ~B1,2 ~tdm1,2->1) [[3]] - (E[0, 0, y1 a2] ~B1,2 ~tdm1,2->1) [[3]]),
  " [b,x] = "
  ((E[0, 0, x2 b1] ~B1,2 ~tdm1,2->1) [[3]] - (E[0, 0, x1 b2] ~B1,2 ~tdm1,2->1) [[3]]), " xy - qyx = "
  ((E[0, 0, x1 y2] ~B1,2 ~tdm1,2->1) [[3]] - (1 + e) (E[0, 0, y1 x2] ~B1,2 ~tdm1,2->1) [[3]])
} /. {z_1 -> z, e -> E} // Expand // Factor
{
  " Δ(a) = " ((E[0, 0, a1] ~B1 ~tdΔ1->1,2) [[3]]),
  " Δ(x) = " ((E[0, 0, x1] ~B1 ~tdΔ1->1,2) [[3]]),
  " Δ(b) = " ((E[0, 0, b1] ~B1 ~tdΔ1->1,2) [[3]]),
  " Δ(y) = " ((E[0, 0, y1] ~B1 ~tdΔ1->1,2) [[3]])
} /. {e -> E} // Simplify
{
  " S(a) = " ((E[0, 0, a1] ~B1 ~tdS1) [[3]]),
  " S(x) = " ((E[0, 0, x1] ~B1 ~tdS1) [[3]]),
  " S(b) = " ((E[0, 0, b1] ~B1 ~tdS1) [[3]]),
  " S(y) = " ((E[0, 0, y1] ~B1 ~tdS1) [[3]])
} /. {z_1 -> z, e -> E} // Simplify

```

```

Out[*]:= {- [a,y] = y, [b,x] = x e, xy - qyx = e^-b (-1 + e^b + a e)}

```

```

Out[*]:= { Δ(a) = (a1 + a2), Δ(x) = (x1 + (1 - e a1) x2), Δ(b) = (b1 + b2), Δ(y) = (y1 + e^-b1 y2) }

```

```

Out[*]:= {- S(a) = a, - S(x) = (x + a x e), - S(b) = b, S(y) = e^b y (-1 + e)}

```

Hopf algebra axioms on double

(co)-associativity

```

In[*]:= { (tdΔ1->1,2 ~B2 ~tdΔ2->2,3) ≡ (tdΔ1->1,3 ~B1 ~tdΔ1->1,2),
  (tdm1,2->1 ~B1 ~tdm1,3->1) ≡ (tdm2,3->2 ~B2 ~tdm1,2->1) }

```

```

Out[*]:= {True, True}

```

Δ is an algebra morphism

```

In[*]:= tdm1,2->1 ~B1 ~tdΔ1->1,2 ≡ (tdΔ1->1,3 tdΔ2->2,4) ~B1,2,3,4 ~ (tdm3,4->2 tdm1,2->1) /. e -> E // Timing

```

```

Out[*]:= {247., True}

```

S is convolution inverse of id

```

In[*]:= { (tdΔ1->1,2 ~B1 ~tdS1) ~B1,2 ~tdm1,2->1, (tdΔ1->1,2 ~B2 ~tdS2) ~B1,2 ~tdm1,2->1 } /. e -> E // Timing

```

```

Out[*]:= {95.136, {E[0, 0, 1], E[0, 0, 1]}}

```

S is a (co)-algebra anti-morphism

```

In[*]:= { tdm1,2->1 ~B1 ~tdS1 ≡ (tdS1 tdS2) ~B1,2 ~tdm2,1->1,
  tdm1,2->1 ~B1 ~tdΔ1->1,2 ≡ tdΔ1->2,1 ~B1,2 ~ (tdS1 tdS2) } /. e -> E // Expand // Timing

```

```

Out[*]:= {151.288, {True, True}}

```

R-matrix

In[*]:= Series[e_{q,1}[z] /. {z → y_i x_j, q → 1 + ρ}, {ρ, 0, 1}] /. {e → E, ρ → ε}

Out[*]=
$$e^{x_j y_i} - \frac{1}{4} (e^{x_j y_i} x_j^2 y_i^2) \in + O[\epsilon]^2$$

In[*]:=
$$e_{q,r}[x_-] := e^{\left(\sum_{j=1}^{k+1} \left((1-q)^j x^j \right) / (j (1-q^j)) \right)}$$

$$tR_{i,j_-} := \mathbb{E} [b_i a_j, y_i x_j, 1 - \epsilon \frac{1}{4} y_i^2 x_j^2]$$
 (*First two terms in Faddeev-Quesne formula*)

Quasi-triangular axiom 1:

In[*]:= tR_{1,2} ~ B₁ ~ tΔ_{1→1,3} ≡ (tR_{1,4} tR_{3,2}) ~ B_{2,4} ~ tdm_{2,4→2} /. {e → E}

Out[*]= True

Quasi-triangular axiom 2:

In[*]:= ((tΔ_{1→1,2} tR_{3,4}) ~ B_{1,2,3,4} ~ (tdm_{1,3→1} tdm_{2,4→2}) /. {e → E}) ≡ ((tΔ_{1→2,1} tR_{3,4}) ~ B_{1,2,3,4} ~ (tdm_{3,1→1} tdm_{4,2→2}) /. {e → E})

Out[*]= True

Reidemeister 3:

In[*]:= ((tR_{1,2} tR_{4,3} tR_{5,6}) ~ B_{1,4} ~ tdm_{1,4→1} ~ B_{2,5} ~ tdm_{2,5→2} ~ B_{3,6} ~ tdm_{3,6→3} /. e → E) ≡ ((tR_{1,6} tR_{2,3} tR_{4,5}) ~ B_{1,4} ~ tdm_{1,4→1} ~ B_{2,5} ~ tdm_{2,5→2} ~ B_{3,6} ~ tdm_{3,6→3} /. e → E) // Timing

Out[*]= {41.732, True}

In[*]:= (*Deriving tR̄ formula*)
Expand /@ tR_{i,j} ~ B_j ~ tΔ_{S_j}

Out[*]=
$$\mathbb{E} \left[-a_j b_i, -e^{b_i} x_j y_i, 1 - e^{b_i} \epsilon a_j x_j y_i - \frac{3}{4} e^{2 b_i} \epsilon x_j^2 y_i^2 \right]$$

In[*]:=
$$\overline{tR}_{i,j_-} := \mathbb{E} \left[-a_j b_i, -e^{b_i} x_j y_i, 1 - e^{b_i} \epsilon a_j x_j y_i - \frac{3}{4} e^{2 b_i} \epsilon x_j^2 y_i^2 \right]$$

Reidemeister 2

In[*]:= { (tR̄_{1,2} tR_{3,4}) ~ B_{1,2,3,4} ~ (tdm_{1,3→1} tdm_{2,4→2}), (tR_{1,2} tR̄_{3,4}) ~ B_{1,2,3,4} ~ (tdm_{1,3→1} tdm_{2,4→2}) }

Out[*]= {E[0, 0, 1], E[0, 0, 1]}

Changing to the central variable t

The full zipping procedure only works when the coefficient functions commute.

We therefore need to introduce the central element t = ε a - b.

$$\begin{aligned} \text{In[*]}:= \quad & \mathbf{b2t}_i := \mathbb{E}[\alpha_i \mathbf{a}_i - \beta_i \mathbf{t}_i, \xi_i \mathbf{x}_i + \eta_i \mathbf{y}_i, \mathbf{1} + \epsilon \beta_i \mathbf{a}_i] \\ & \mathbf{t2b}_i := \mathbb{E}[\alpha_i \mathbf{a}_i - \tau_i \mathbf{b}_i, \xi_i \mathbf{x}_i + \eta_i \mathbf{y}_i, \mathbf{1} + \epsilon \tau_i \mathbf{a}_i] \end{aligned}$$

(*converting the Hopf algebra operations*)

$$\begin{aligned} & (\mathbf{t2b}_i \mathbf{t2b}_j) \sim \mathbf{B}_{i,j} \sim \mathbf{tdm}_{i,j \rightarrow k} \sim \mathbf{B}_k \sim \mathbf{b2t}_k \\ & \mathbf{t2b}_i \sim \mathbf{B}_i \sim \mathbf{td}\Delta_{i \rightarrow j,k} \sim \mathbf{B}_{j,k} \sim (\mathbf{b2t}_j \mathbf{b2t}_k) / . \mathbf{e} \rightarrow \mathbf{E} \\ & \mathbf{t2b}_i \sim \mathbf{B}_i \sim \mathbf{tdS}_i \sim \mathbf{B}_i \sim \mathbf{b2t}_i \end{aligned}$$

$$\begin{aligned} \text{Out[*]}:= \quad & \mathbb{E} \left[\mathbf{a}_k \alpha_i + \mathbf{a}_k \alpha_j + \mathbf{t}_k \tau_i + \mathbf{t}_k \tau_j, \right. \\ & e^{-\alpha_i - \alpha_j} \left(e^{\alpha_i + \alpha_j} \mathbf{y}_k \eta_i + e^{\alpha_j} \mathbf{y}_k \eta_j + e^{\alpha_i} \mathbf{x}_k \xi_i + e^{\alpha_i + \alpha_j} \eta_j \xi_i - e^{\mathbf{t}_k + \alpha_i + \alpha_j} \eta_j \xi_i + e^{\alpha_i + \alpha_j} \mathbf{x}_k \xi_j \right), \\ & \frac{1}{4} e^{-\alpha_i - \alpha_j} \left(4 e^{\alpha_i + \alpha_j} + 8 e^{\mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \mathbf{a}_k \eta_j \xi_i + 4 \epsilon \mathbf{x}_k \mathbf{y}_k \eta_j \xi_i + 2 e^{\alpha_j} \epsilon \mathbf{y}_k \eta_j^2 \xi_i - 6 e^{\mathbf{t}_k + \alpha_j} \epsilon \mathbf{y}_k \eta_j^2 \xi_i + \right. \\ & \left. 2 e^{\alpha_i} \epsilon \mathbf{x}_k \eta_j \xi_i^2 - 6 e^{\mathbf{t}_k + \alpha_i} \epsilon \mathbf{x}_k \eta_j \xi_i^2 + e^{\alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 - 4 e^{\mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 + 3 e^{2 \mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 \right) \end{aligned}$$

$$\begin{aligned} \text{Out[*]}:= \quad & \mathbb{E} \left[\mathbf{a}_j \alpha_i + \mathbf{a}_k \alpha_i + \mathbf{t}_j \tau_i + \mathbf{t}_k \tau_i, \mathbf{y}_j \eta_i + e^{\mathbf{t}_j} \mathbf{y}_k \eta_i + \mathbf{x}_j \xi_i + \mathbf{x}_k \xi_i, \right. \\ & \left. \frac{1}{2} \left(2 - 2 e^{\mathbf{t}_j} \epsilon \mathbf{a}_j \mathbf{y}_k \eta_i + e^{\mathbf{t}_j} \epsilon \mathbf{y}_j \mathbf{y}_k \eta_i^2 - 2 \epsilon \mathbf{a}_j \mathbf{x}_k \xi_i + \epsilon \mathbf{x}_j \mathbf{x}_k \xi_i^2 \right) \right] \end{aligned}$$

$$\begin{aligned} \text{Out[*]}:= \quad & \mathbb{E} \left[-\mathbf{a}_i \alpha_i - \mathbf{t}_i \tau_i, e^{-\mathbf{t}_i} \left(-e^{\alpha_i} \mathbf{y}_i \eta_i - e^{\mathbf{t}_i + \alpha_i} \mathbf{x}_i \xi_i + e^{\alpha_i} \eta_i \xi_i - e^{\mathbf{t}_i + \alpha_i} \eta_i \xi_i \right), \right. \\ & \frac{1}{4} e^{-2 \mathbf{t}_i} \left(4 e^{2 \mathbf{t}_i} + 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{y}_i \eta_i - 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \mathbf{y}_i \eta_i - 2 e^{2 \alpha_i} \epsilon \mathbf{y}_i^2 \eta_i^2 - 4 e^{2 \mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \mathbf{x}_i \xi_i - \right. \\ & 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \eta_i \xi_i + 4 e^{2 \mathbf{t}_i + \alpha_i} \epsilon \eta_i \xi_i + 8 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \eta_i \xi_i - 4 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \mathbf{y}_i \eta_i \xi_i + \\ & 6 e^{2 \alpha_i} \epsilon \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i^2 \xi_i^2 + 6 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \eta_i \xi_i^2 - \\ & \left. \left. 2 e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \eta_i \xi_i^2 - 3 e^{2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 + 4 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 - e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 \right) \right] \end{aligned}$$

$$\begin{aligned} \text{In[*]}:= \quad & \mathbf{dm}_{i,j \rightarrow k} := \mathbb{E} \left[\mathbf{a}_k \alpha_i + \mathbf{a}_k \alpha_j + \mathbf{t}_k \tau_i + \mathbf{t}_k \tau_j, \right. \\ & e^{-\alpha_i - \alpha_j} \left(e^{\alpha_i + \alpha_j} \mathbf{y}_k \eta_i + e^{\alpha_j} \mathbf{y}_k \eta_j + e^{\alpha_i} \mathbf{x}_k \xi_i + e^{\alpha_i + \alpha_j} \eta_j \xi_i - e^{\mathbf{t}_k + \alpha_i + \alpha_j} \eta_j \xi_i + e^{\alpha_i + \alpha_j} \mathbf{x}_k \xi_j \right), \\ & \frac{1}{4} e^{-\alpha_i - \alpha_j} \left(4 e^{\alpha_i + \alpha_j} + 8 e^{\mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \mathbf{a}_k \eta_j \xi_i + 4 \epsilon \mathbf{x}_k \mathbf{y}_k \eta_j \xi_i + 2 e^{\alpha_j} \epsilon \mathbf{y}_k \eta_j^2 \xi_i - 6 e^{\mathbf{t}_k + \alpha_j} \epsilon \mathbf{y}_k \eta_j^2 \xi_i + \right. \\ & \left. 2 e^{\alpha_i} \epsilon \mathbf{x}_k \eta_j \xi_i^2 - 6 e^{\mathbf{t}_k + \alpha_i} \epsilon \mathbf{x}_k \eta_j \xi_i^2 + e^{\alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 - 4 e^{\mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 + 3 e^{2 \mathbf{t}_k + \alpha_i + \alpha_j} \epsilon \eta_j^2 \xi_i^2 \right) \end{aligned}$$

$$\begin{aligned} \mathbf{d}\Delta_{i \rightarrow j,k} := \quad & \mathbb{E} \left[\mathbf{a}_j \alpha_i + \mathbf{a}_k \alpha_i + \mathbf{t}_j \tau_i + \mathbf{t}_k \tau_i, \mathbf{y}_j \eta_i + e^{\mathbf{t}_j} \mathbf{y}_k \eta_i + \mathbf{x}_j \xi_i + \mathbf{x}_k \xi_i, \right. \\ & \left. \frac{1}{2} \left(2 - 2 e^{\mathbf{t}_j} \epsilon \mathbf{a}_j \mathbf{y}_k \eta_i + e^{\mathbf{t}_j} \epsilon \mathbf{y}_j \mathbf{y}_k \eta_i^2 - 2 \epsilon \mathbf{a}_j \mathbf{x}_k \xi_i + \epsilon \mathbf{x}_j \mathbf{x}_k \xi_i^2 \right) \right] \end{aligned}$$

$$\begin{aligned} \mathbf{dS}_i := \quad & \mathbb{E} \left[-\mathbf{a}_i \alpha_i - \mathbf{t}_i \tau_i, e^{-\mathbf{t}_i} \left(-e^{\alpha_i} \mathbf{y}_i \eta_i - e^{\mathbf{t}_i + \alpha_i} \mathbf{x}_i \xi_i + e^{\alpha_i} \eta_i \xi_i - e^{\mathbf{t}_i + \alpha_i} \eta_i \xi_i \right), \right. \\ & \frac{1}{4} e^{-2 \mathbf{t}_i} \left(4 e^{2 \mathbf{t}_i} + 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{y}_i \eta_i - 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \mathbf{y}_i \eta_i - 2 e^{2 \alpha_i} \epsilon \mathbf{y}_i^2 \eta_i^2 - 4 e^{2 \mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \mathbf{x}_i \xi_i - \right. \\ & 4 e^{\mathbf{t}_i + \alpha_i} \epsilon \eta_i \xi_i + 4 e^{2 \mathbf{t}_i + \alpha_i} \epsilon \eta_i \xi_i + 8 e^{\mathbf{t}_i + \alpha_i} \epsilon \mathbf{a}_i \eta_i \xi_i - 4 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \mathbf{y}_i \eta_i \xi_i + \\ & 6 e^{2 \alpha_i} \epsilon \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{y}_i \eta_i^2 \xi_i - 2 e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i^2 \xi_i^2 + 6 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \eta_i \xi_i^2 - \\ & \left. \left. 2 e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \mathbf{x}_i \eta_i \xi_i^2 - 3 e^{2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 + 4 e^{\mathbf{t}_i + 2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 - e^{2 \mathbf{t}_i + 2 \alpha_i} \epsilon \eta_i^2 \xi_i^2 \right) \right] \end{aligned}$$

Does it match the m-tensor from SLPortfolioProgram.nb? They do match on the generators?


```

In[*]:= {
  " [t,y] = " ((E[0, 0, y2 t1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, y1 t2] ~ B1,2 ~ dm1,2->1) [[3]]),
  " [t,a] = " ((E[0, 0, a2 t1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, a1 t2] ~ B1,2 ~ dm1,2->1) [[3]]),
  " [t,x] = " ((E[0, 0, x2 t1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, x1 t2] ~ B1,2 ~ dm1,2->1) [[3]]),
  " [a,y] = " ((E[0, 0, y2 a1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, y1 a2] ~ B1,2 ~ dm1,2->1) [[3]]),
  " [x,a] = " ((E[0, 0, a2 x1] ~ B1,2 ~ dm1,2->1) [[3]] - (E[0, 0, a1 x2] ~ B1,2 ~ dm1,2->1) [[3]]),
  " xy - qyx = "
  ((E[0, 0, x1 y2] ~ B1,2 ~ dm1,2->1) [[3]] - (1 + e) (E[0, 0, y1 x2] ~ B1,2 ~ dm1,2->1) [[3]])
} /. {z_1 -> z, e -> E} // Expand // Factor
{
  " Δ(a) = " ((E[0, 0, a1] ~ B1 ~ dΔ1->1,2) [[3]]),
  " Δ(x) = " ((E[0, 0, x1] ~ B1 ~ dΔ1->1,2) [[3]]),
  " Δ(t) = " ((E[0, 0, t1] ~ B1 ~ dΔ1->1,2) [[3]]),
  " Δ(y) = " ((E[0, 0, y1] ~ B1 ~ dΔ1->1,2) [[3]])
} /. {e -> E} // Simplify
{
  " S(a) = " ((E[0, 0, a1] ~ B1 ~ dS1) [[3]]),
  " S(x) = " ((E[0, 0, x1] ~ B1 ~ dS1) [[3]]),
  " S(t) = " ((E[0, 0, t1] ~ B1 ~ dS1) [[3]]),
  " S(y) = " ((E[0, 0, y1] ~ B1 ~ dS1) [[3]])
} /. {z_1 -> z, e -> E} // Simplify

```

```
Out[*]:= {0, 0, 0, - [a,y] = y, - [x,a] = x, xy - qyx = (1 - e^t + 2 a e^t e)}
```

```
Out[*]:= { Δ(a) = (a1 + a2), Δ(x) = (x1 + (1 - e a1) x2),
  Δ(t) = (t1 + t2), Δ(y) = (y1 - e^t1 (-1 + e a1) y2) }
```

```
Out[*]:= {- S(a) = a, - S(x) = (x + a x e), - S(t) = t, - S(y) = e^-t y (1 + (-1 + a) e)}
```

Hopf algebra axioms on double

(co)-associativity

```

In[*]:= { (dΔ1->1,2 ~ B2 ~ dΔ2->2,3) ≡ (dΔ1->1,3 ~ B1 ~ dΔ1->1,2),
  (dm1,2->1 ~ B1 ~ dm1,3->1) ≡ (dm2,3->2 ~ B2 ~ dm1,2->1) }

```

```
Out[*]:= {True, True}
```

Δ is an algebra morphism

```
In[*]:= dm1,2->1 ~ B1 ~ dΔ1->1,2 ≡ (dΔ1->1,3 dΔ2->2,4) ~ B1,2,3,4 ~ (dm3,4->2 dm1,2->1) /. e -> E // Timing
```

```
Out[*]:= {105.488, True}
```

S is convolution inverse of id

```
In[*]:= { (dΔ1->1,2 ~ B1 ~ dS1) ~ B1,2 ~ dm1,2->1, (dΔ1->1,2 ~ B2 ~ dS2) ~ B1,2 ~ dm1,2->1 } /. e -> E // Timing
```

```
Out[*]:= {34.684, {E[0, 0, 1], E[0, 0, 1]}}
```

S is a (co)-algebra anti-morphism

In[*]:= $\{dm_{1,2 \rightarrow 1} \sim B_1 \sim dS_1 \equiv (dS_1 dS_2) \sim B_{1,2} \sim dm_{2,1 \rightarrow 1}, dS_1 \sim B_1 \sim d\Delta_{1 \rightarrow 1,2} \equiv d\Delta_{1 \rightarrow 2,1} \sim B_{1,2} \sim (dS_1 dS_2)\} /. e \rightarrow E //$
Expand // Timing

Out[*]:= {133.812, {True, True}}

R-matrix

In[*]:= $tR_{i,j} \sim B_{i,j} \sim (b_2 t_i b_2 t_j)$
 $\bar{t}R_{i,j} \sim B_{i,j} \sim (b_2 t_i b_2 t_j)$

Out[*]:= $E[-a_j t_i, x_j y_i, \frac{1}{4} (4 + 4 \epsilon a_i a_j - \epsilon x_j^2 y_i^2)]$

Out[*]:= $E[a_j t_i, -e^{-t_i} x_j y_i, \frac{1}{4} e^{-2 t_i} (4 e^{2 t_i} - 4 e^{2 t_i} \epsilon a_i a_j - 4 e^{t_i} \epsilon a_i x_j y_i - 4 e^{t_i} \epsilon a_j x_j y_i - 3 \epsilon x_j^2 y_i^2)]$

In[*]:= **(*Deriving \bar{R} formula*)**
Expand /@ $R_{i,j} \sim B_j \sim dS_j$

Out[*]:= $E[a_j t_i, -e^{-t_i} x_j y_i, 1 - \epsilon a_i a_j - e^{-t_i} \epsilon a_i x_j y_i - e^{-t_i} \epsilon a_j x_j y_i - \frac{3}{4} e^{-2 t_i} \epsilon x_j^2 y_i^2]$

```
R_{i_,j_} := E[-a_j t_i, x_j y_i, 1/4 (4 + 4 \epsilon a_i a_j - \epsilon x_j^2 y_i^2)]
\bar{R}_{i_,j_} :=
E[a_j t_i, -e^{-t_i} x_j y_i, 1/4 e^{-2 t_i} (4 e^{2 t_i} - 4 e^{2 t_i} \epsilon a_i a_j - 4 e^{t_i} \epsilon a_i x_j y_i - 4 e^{t_i} \epsilon a_j x_j y_i - 3 \epsilon x_j^2 y_i^2)]
```

Quasi-triangular axiom 1:

In[*]:= $R_{1,2} \sim B_1 \sim d\Delta_{1 \rightarrow 1,3} \equiv (R_{1,4} R_{3,2}) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2}$

Out[*]:= True

Quasi-triangular axiom 2:

In[*]:= $(d\Delta_{1 \rightarrow 1,2} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2}) \equiv (d\Delta_{1 \rightarrow 2,1} R_{3,4}) \sim B_{1,2,3,4} \sim (dm_{3,1 \rightarrow 1} dm_{4,2 \rightarrow 2})$

Out[*]:= True

Reidemeister 3:

In[*]:= $((R_{1,2} R_{4,3} R_{5,6}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3} \equiv (R_{1,6} R_{2,3} R_{4,5}) \sim B_{1,4} \sim dm_{1,4 \rightarrow 1} \sim B_{2,5} \sim dm_{2,5 \rightarrow 2} \sim B_{3,6} \sim dm_{3,6 \rightarrow 3}) /. e \rightarrow E //$
Timing

Out[*]:= {36.072, True}

Deriving the Drinfeld element u and its inverse ui

```
u_{i_} := R_{1,2} \sim B_1 \sim dS_1 \sim B_{1,2} \sim dm_{2,1 \rightarrow i}
ui_{i_} := R_{1,2} \sim B_2 \sim dS_2 \sim B_2 \sim dS_2 \sim B_{1,2} \sim dm_{2,1 \rightarrow i}
```

$$\text{In[*]} := \{ \mathbf{u}_1, \mathbf{u}_1 \}$$

$$\text{Out[*]} := \left\{ \mathbb{E} \left[\mathbf{a}_1 \mathbf{t}_1, -e^{-\mathbf{t}_1} x_1 y_1, \frac{1}{4} e^{-\mathbf{t}_1} \left(4 e^{2 \mathbf{t}_1} - 8 e^{2 \mathbf{t}_1} \in \mathbf{a}_1 - 4 e^{2 \mathbf{t}_1} \in \mathbf{a}_1^2 - 4 e^{\mathbf{t}_1} \in x_1 y_1 - 8 e^{\mathbf{t}_1} \in \mathbf{a}_1 x_1 y_1 - 3 \in x_1^2 y_1^2 \right) \right], \mathbb{E} \left[-\mathbf{a}_1 \mathbf{t}_1, x_1 y_1, \frac{1}{4} e^{-\mathbf{t}_1} \left(4 + 8 \in \mathbf{a}_1 + 4 \in \mathbf{a}_1^2 - 4 \in x_1 y_1 - \in x_1^2 y_1^2 \right) \right] \right\}$$

u and ui are inverses

$$\text{In[*]} := (\mathbf{u}_1 \mathbf{u}_1) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1}$$

$$\text{Out[*]} := \mathbb{E} [\mathbf{0}, \mathbf{0}, 1]$$

The ribbon element v satisfies $v^2 = S(u) u$. The spinner $C = uv^{-1}$.

It is convenient to compute $z = S(u) u^{-1}$ which is something easy.

$$\text{In[*]} := ((\mathbf{u}_1 \sim \mathbf{B}_1 \sim \mathbf{dS}_1) \mathbf{u}_1) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1}$$

$$\text{Out[*]} := \mathbb{E} [\mathbf{0}, \mathbf{0}, e^{-\mathbf{t}_1} (1 + 2 \in \mathbf{a}_1)]$$

So in our case $S(u) = u z$ so $S(u)u = u^2 z$ and $v = uz^{\frac{1}{2}}$ and finally $C = uv^{-1} = z^{-\frac{1}{2}} = e^{\mathbf{t}_1/2} (1 - \in \mathbf{a}_1)$.

$$\mathbf{CC}_{i_} := \mathbb{E} [\mathbf{0}, \mathbf{0}, e^{\frac{1}{2} \mathbf{t}_i} (1 - \in \mathbf{a}_i)]$$

$$\overline{\mathbf{CC}}_{i_} := \mathbb{E} [\mathbf{0}, \mathbf{0}, e^{-\frac{1}{2} \mathbf{t}_i} (1 + \in \mathbf{a}_i)]$$

Kinks

$$\text{In[*]} := \mathbf{k1} = (\mathbf{R}_{1,3} \overline{\mathbf{CC}}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow i} / . \mathbf{e} \rightarrow \mathbf{E};$$

$$\mathbf{k2} = (\mathbf{R}_{3,1} \mathbf{CC}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow i} / . \mathbf{e} \rightarrow \mathbf{E};$$

$$\mathbf{k3} = (\overline{\mathbf{R}}_{1,3} \mathbf{CC}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow j} / . \mathbf{e} \rightarrow \mathbf{E};$$

$$\mathbf{k4} = (\overline{\mathbf{R}}_{3,1} \overline{\mathbf{CC}}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{dm}_{1,2 \rightarrow 1} \sim \mathbf{B}_{1,3} \sim \mathbf{dm}_{1,3 \rightarrow j} / . \mathbf{e} \rightarrow \mathbf{E};$$

$$\{ \mathbf{k1} \equiv \mathbf{k2}, \mathbf{k3} \equiv \mathbf{k4}, (\mathbf{k1} \mathbf{k3}) \sim \mathbf{B}_{i,j} \sim \mathbf{dm}_{i,j \rightarrow 1} \}$$

k1

k3

$$\text{Out[*]} := \{ \text{True}, \text{True}, \mathbb{E} [\mathbf{0}, \mathbf{0}, 1] \}$$

$$\text{Out[*]} := \mathbb{E} [-\mathbf{a}_i \mathbf{t}_i, x_i y_i, \frac{1}{4} e^{-\frac{\mathbf{t}_i}{2}} (4 + 4 \in \mathbf{a}_i + 4 \in \mathbf{a}_i^2 - \in x_i^2 y_i^2)]$$

$$\text{Out[*]} := \mathbb{E} [\mathbf{a}_j \mathbf{t}_j, -e^{-\mathbf{t}_j} x_j y_j, \frac{1}{4} e^{-2 \mathbf{t}_j} \left(4 e^{\frac{5 \mathbf{t}_j}{2}} - 4 e^{\frac{5 \mathbf{t}_j}{2}} \in \mathbf{a}_j - 4 e^{\frac{5 \mathbf{t}_j}{2}} \in \mathbf{a}_j^2 - 8 e^{\frac{3 \mathbf{t}_j}{2}} \in \mathbf{a}_j x_j y_j - 3 e^{\frac{\mathbf{t}_j}{2}} \in x_j^2 y_j^2 \right)]$$

$$\text{In[*]} := \mathbf{Kink}_{i_} := \mathbb{E} [-\mathbf{a}_i \mathbf{t}_i, x_i y_i, \frac{1}{4} e^{-\frac{\mathbf{t}_i}{2}} (4 + 4 \in \mathbf{a}_i + 4 \in \mathbf{a}_i^2 - \in x_i^2 y_i^2)]$$

$$\overline{\mathbf{Kink}}_{j_} :=$$

$$\mathbb{E} [\mathbf{a}_j \mathbf{t}_j, -e^{-\mathbf{t}_j} x_j y_j, \frac{1}{4} e^{-2 \mathbf{t}_j} \left(4 e^{\frac{5 \mathbf{t}_j}{2}} - 4 e^{\frac{5 \mathbf{t}_j}{2}} \in \mathbf{a}_j - 4 e^{\frac{5 \mathbf{t}_j}{2}} \in \mathbf{a}_j^2 - 8 e^{\frac{3 \mathbf{t}_j}{2}} \in \mathbf{a}_j x_j y_j - 3 e^{\frac{\mathbf{t}_j}{2}} \in x_j^2 y_j^2 \right)]$$

Reidemeister 2:

In[*]:= $(R_{1,2} \overline{R}_{3,4}) \sim B_{1,3} \sim dm_{1,3 \rightarrow 1} \sim B_{2,4} \sim dm_{2,4 \rightarrow 2}$

Out[*]:= $\mathbb{E}[\emptyset, \emptyset, 1]$

cyclic Reidemeister 2:

In[*]:= $(R_{1,4} \overline{R}_{5,2} \overline{CC}_3) \sim B_{2,4} \sim dm_{2,4 \rightarrow 2} \sim B_{1,3} \sim dm_{1,3 \rightarrow 1} \sim B_{1,5} \sim dm_{1,5 \rightarrow 1} \equiv \overline{CC}_1$

Out[*]:= True

Trefoil

In[*]:= $Z = R_{1,5} R_{6,2} R_{3,7} \overline{CC}_4 \overline{Kink}_8 \overline{Kink}_9 \overline{Kink}_{10};$

Do[Print["doing ", r]; Z = Z ~ B_{1,r} ~ dm_{1,r→1} /. e → E, {r, 2, 10}] // Timing

Z

1 / (Z[[3]] /. e^{c·t₁} → T^c) // Expand

doing 2

doing 3

doing 4

doing 5

doing 6

doing 7

doing 8

doing 9

doing 10

Out[*]:= {867.924, Null}

Out[*]:= $\mathbb{E}[\emptyset, \emptyset, (e^{t_1} - 2 e^{2 t_1} + 3 e^{3 t_1} - 2 e^{4 t_1} + e^{5 t_1} - e^{2 t_1} \epsilon + 2 e^{3 t_1} \epsilon - 3 e^{4 t_1} \epsilon + 2 e^{5 t_1} \epsilon - 2 e^{t_1} \epsilon a_1 + 2 e^{2 t_1} \epsilon a_1 - 2 e^{4 t_1} \epsilon a_1 + 2 e^{5 t_1} \epsilon a_1 - 2 e^{t_1} \epsilon x_1 y_1 - 2 e^{4 t_1} \epsilon x_1 y_1) / (1 - 3 e^{t_1} + 6 e^{2 t_1} - 7 e^{3 t_1} + 6 e^{4 t_1} - 3 e^{5 t_1} + e^{6 t_1})]$

In[*]:= **1 / Z[[3]] /. {e^{c·t₁} → T^c, ε → 0} // Factor // Expand**

Coefficient[Z[[3]] /. e^{c·t₁} → T^c, ε] // Factor

Out[*]:= $-1 + \frac{1}{T} + T$

Out[*]:= $(T (-T + 2 T^2 - 3 T^3 + 2 T^4 - 2 a_1 + 2 T a_1 - 2 T^3 a_1 + 2 T^4 a_1 - 2 x_1 y_1 - 2 T^3 x_1 y_1)) / (1 - T + T^2)^3$

In[*]:= **Z[[3]] /. e^{c·t₁} → T^c // Factor**

Out[*]:= $(T (1 - 2 T + 3 T^2 - 2 T^3 + T^4 - T \epsilon + 2 T^2 \epsilon - 3 T^3 \epsilon + 2 T^4 \epsilon - 2 \epsilon a_1 + 2 T \epsilon a_1 - 2 T^3 \epsilon a_1 + 2 T^4 \epsilon a_1 - 2 \epsilon x_1 y_1 - 2 T^3 \epsilon x_1 y_1)) / (1 - T + T^2)^3$