

## The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues MinimalGassnerPlus.nb and pensieve://Projects/OneCo-1604/, continued pensieve://2019-11/.

```
In[ ]:= Kδ /: Kδis := KroneckerDelta[1, Length[Union[{is}]]];
```

### The Burau Representation

```
In[ ]:= Bi,j[ξ] := ξ /. vj => (1 - t) vi + t vj

In[ ]:= Column@{R3l = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
  R3r = {v1, v2, v3} // B2,3 // B1,3 // B1,2,
  R3l - R3r // Expand}

Out[ ]:= {v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
{v1, (1 - t) v1 + t v2, (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
{0, 0, 0}
```

### The Gassner Representation

```
Gi,j[ξ] := ξ /. vj => (1 - ti) vi + ti vj

Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
  R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2,
  R3l - R3r // Expand}

{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, 0}

Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
  OCr = {v1, v2, v3} // G1,3 // G1,2,
  OCl - OCr // Expand}

{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{0, 0, 0}

Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
  UCr = {v1, v2, v3} // G2,3 // G1,3,
  UCl - UCr // Expand}

{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

## The Gassner-Plus Representation

$GP_{i,j}[\underline{\xi}] := \text{Expand}[\underline{\xi} /. \{u_j \rightarrow (1 - t_i) u_i + t_i u_j, \\ f_{-} \cdot v_j \rightarrow f(1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i \partial_{t_i} f - t_j \partial_{t_j} f) u_i + f t_i u_i \}];$

$GP\text{checks} = \{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3\};$

$R3l = GP\text{checks} // GP_{1,2} // GP_{1,3} // GP_{2,3}$

$\{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 - \\ f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 + t_2 u_1 f^{(0,1,0)}[t_1, t_2, t_3] - \\ t_1 t_2 u_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 u_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 u_1 f^{(1,0,0)}[t_1, t_2, t_3], \\ f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] t_1 t_2 u_2 + f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \\ f[t_1, t_2, t_3] t_1 v_2 - f[t_1, t_2, t_3] t_1 t_2 v_2 + f[t_1, t_2, t_3] t_1 t_2 v_3 + t_3 u_1 f^{(0,0,1)}[t_1, t_2, t_3] - \\ t_1 t_3 u_1 f^{(0,0,1)}[t_1, t_2, t_3] + t_1 t_3 u_2 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_2 t_3 u_2 f^{(0,0,1)}[t_1, t_2, t_3] - \\ t_1 t_2 u_2 f^{(0,1,0)}[t_1, t_2, t_3] + t_1 t_2^2 u_2 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 u_1 f^{(1,0,0)}[t_1, t_2, t_3] + \\ t_1^2 u_1 f^{(1,0,0)}[t_1, t_2, t_3], u_1, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_2 - t_1 t_2 u_2 + t_1 t_2 u_3\}$

$R3r = GP\text{checks} // GP_{2,3} // GP_{1,3} // GP_{1,2}; R3l - R3r$

$\{0, 0, 0, 0, 0, 0\}$

$OC1 = GP\text{checks} // GP_{1,2} // GP_{1,3}$

$\{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 - \\ f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 + t_2 u_1 f^{(0,1,0)}[t_1, t_2, t_3] - \\ t_1 t_2 u_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 u_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 u_1 f^{(1,0,0)}[t_1, t_2, t_3], \\ f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_3 + \\ t_3 u_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_3 u_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 u_1 f^{(1,0,0)}[t_1, t_2, t_3] + \\ t_1^2 u_1 f^{(1,0,0)}[t_1, t_2, t_3], u_1, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_3\}$

$OCr = GP\text{checks} // GP_{1,3} // GP_{1,2}; OC1 - OCr$

$\{0, 0, 0, 0, 0, 0\}$

Question. Does GP factor through G? How?

## The End(G) Representation

$EG_{i,j}[\underline{\xi}] := \text{Expand}[\underline{\xi} /. \{u_j \rightarrow (1 - t_i) u_i + t_i u_j, w_i \rightarrow w_i + (1 - t_i^{-1}) w_j, w_j \rightarrow t_i^{-1} w_j\}];$   
 $EG\text{checks} = \text{Flatten@Table}[u_i w_j, \{i, 3\}, \{j, 3\}]$

$\{u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3\}$

Short [R3l = EGchecks // EG<sub>1,2</sub> // EG<sub>1,3</sub> // EG<sub>2,3</sub>, 10]

$$\left\{ u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1}, \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2}, \frac{u_1 w_3}{t_1 t_2}, \right.$$

$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$

$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3, -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$

$$- \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2}, u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$

$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$

$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$

$$- u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$

$$\left. \frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3, - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3 \right\}$$

R3r = EGchecks // EG<sub>2,3</sub> // EG<sub>1,3</sub> // EG<sub>1,2</sub>; R3l - R3r

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

(# → Collect[EG<sub>i,j</sub>[#], u\_w\_, Simplify]) & /@ {u<sub>k</sub> w<sub>j</sub>, u<sub>k</sub> w<sub>i</sub>, u<sub>j</sub> w<sub>k</sub>, u<sub>j</sub> w<sub>i</sub>}

$$\left\{ u_k w_j \rightarrow \frac{u_k w_j}{t_i}, u_k w_i \rightarrow u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j, u_j w_k \rightarrow (1 - t_i) u_i w_k + t_i u_j w_k, \right.$$

$$\left. u_j w_i \rightarrow (1 - t_i) u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i) u_j w_j \right\}$$

## The End(G)+c Representation

Is there topology behind this representation?

$$\text{EGC}_{i,j}[\mathcal{E}] := \text{Expand}[\mathcal{E} /. \{$$

$$c_i \rightarrow c_i - (1 - t_i^{-1}) u_i w_j, c_j \rightarrow c_j + (1 - t_i^{-1}) u_i w_j,$$

$$u_j \rightarrow (1 - t_i) u_i + t_i u_j,$$

$$w_i \rightarrow w_i + (1 - t_i^{-1}) w_j, w_j \rightarrow t_i^{-1} w_j \}];$$

EGchecks = {c<sub>1</sub>, c<sub>2</sub>, c<sub>3</sub>, u<sub>1</sub> w<sub>1</sub>, u<sub>1</sub> w<sub>2</sub>, u<sub>1</sub> w<sub>3</sub>, u<sub>2</sub> w<sub>1</sub>, u<sub>2</sub> w<sub>2</sub>, u<sub>2</sub> w<sub>3</sub>, u<sub>3</sub> w<sub>1</sub>, u<sub>3</sub> w<sub>2</sub>, u<sub>3</sub> w<sub>3</sub>};

(# → Collect[EG<sub>c,1,2</sub>[#], u\_w\_, Simplify]) & /@ EGchecks

$$\left\{ c_1 \rightarrow c_1 + \left(-1 + \frac{1}{t_1}\right) u_1 w_2, c_2 \rightarrow c_2 + \left(1 - \frac{1}{t_1}\right) u_1 w_2, \right.$$

$$c_3 \rightarrow c_3, u_1 w_1 \rightarrow u_1 w_1 + \left(1 - \frac{1}{t_1}\right) u_1 w_2, u_1 w_2 \rightarrow \frac{u_1 w_2}{t_1}, u_1 w_3 \rightarrow u_1 w_3,$$

$$u_2 w_1 \rightarrow (1 - t_1) u_1 w_1 + t_1 u_2 w_1 - \frac{(-1 + t_1)^2 u_1 w_2}{t_1} + (-1 + t_1) u_2 w_2, u_2 w_2 \rightarrow \left(-1 + \frac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$$

$$\left. u_2 w_3 \rightarrow (1 - t_1) u_1 w_3 + t_1 u_2 w_3, u_3 w_1 \rightarrow u_3 w_1 + \left(1 - \frac{1}{t_1}\right) u_3 w_2, u_3 w_2 \rightarrow \frac{u_3 w_2}{t_1}, u_3 w_3 \rightarrow u_3 w_3 \right\}$$

$u_1 w_2 + c_2 // \text{EGC}_{1,2}$

$c_2 + u_1 w_2$

$(\# \rightarrow \text{Simplify}[\text{EGC}_{1,2}[\#] /. \{u_i w_i \rightarrow 1, u_i w_j /; i \neq j \rightarrow 0\}]) \& /@ \text{EGCchecks}$

$\{c_1 \rightarrow c_1, c_2 \rightarrow c_2, c_3 \rightarrow c_3, u_1 w_1 \rightarrow 1, u_1 w_2 \rightarrow 0, u_1 w_3 \rightarrow 0,$   
 $u_2 w_1 \rightarrow 0, u_2 w_2 \rightarrow 1, u_2 w_3 \rightarrow 0, u_3 w_1 \rightarrow 0, u_3 w_2 \rightarrow 0, u_3 w_3 \rightarrow 1\}$

**Short[R31 = EGCchecks // EGC<sub>1,2</sub> // EGC<sub>1,3</sub> // EGC<sub>2,3</sub>, 10]**

$$\left\{ c_1 - u_1 w_2 + \frac{u_1 w_2}{t_1} - u_1 w_3 + \frac{u_1 w_3}{t_1}, c_2 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2}, \right.$$

$$c_3 + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2}, u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1}, \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},$$

$$\frac{u_1 w_3}{t_1 t_2}, u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$

$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3, -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$

$$- \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2}, u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$

$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$

$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$

$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$

$$\left. \frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3, - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3 \right\}$$

**R3r = EGCchecks // EGC<sub>2,3</sub> // EGC<sub>1,3</sub> // EGC<sub>1,2</sub>; R31 - R3r**

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

## The Turbo-Gassner Representation

$\text{TG}_{i,j}[\xi] := \text{Expand}[\xi /. \{$   
 $f_{\cdot} v_{\hat{k}} \rightarrow \text{Plus}[f v_{\hat{k}} /. v_j \rightarrow (1 - t_i) v_i + t_i v_j,$   
 $(1 - t_i^{-1}) (t_i \partial_{t_i} f - t_j \partial_{t_j} f) (u_{\hat{k}} /. u_j \rightarrow (1 - t_i) u_i + t_i u_j) u_i w_j,$   
 $K \delta_{\hat{k}, i} f (u_j - u_i) u_i w_j],$   
 $u_j \rightarrow (1 - t_i) u_i + t_i u_j,$   
 $w_i \rightarrow w_i + (1 - t_i^{-1}) w_j, w_j \rightarrow t_i^{-1} w_j \}];$   
 $\text{TGchecks} = \{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3, w_1, w_2, w_3\};$

**Short [R3l = TGchecks // TG<sub>1,2</sub> // TG<sub>1,3</sub> // TG<sub>2,3</sub>, 10]**

$$\left\{ f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] u_1^2 w_2 + f[t_1, t_2, t_3] u_1 u_2 w_2 - f[t_1, t_2, t_3] u_1^2 w_3 + \right. \\ \left. f[t_1, t_2, t_3] u_1 u_3 w_3 - \frac{t_3 u_1^2 w_3 f^{(0,0,1)}[t_1, t_2, t_3]}{t_2} + \frac{t_3 u_1^2 w_3 f^{(0,0,1)}[t_1, t_2, t_3]}{t_1 t_2} - \right. \\ \left. t_3 u_1 u_2 w_3 f^{(0,0,1)}[t_1, t_2, t_3] + \langle\langle 9 \rangle\rangle + \frac{t_2 u_1^2 w_3 f^{(0,1,0)}[t_1, t_2, t_3]}{t_1} - u_1 u_2 w_3 f^{(0,1,0)}[t_1, t_2, t_3] + \right. \\ \left. t_2 u_1 u_2 w_3 f^{(0,1,0)}[t_1, t_2, t_3] - u_1^2 w_2 f^{(1,0,0)}[t_1, t_2, t_3] + t_1 u_1^2 w_2 f^{(1,0,0)}[t_1, t_2, t_3] - \right. \\ \left. u_1^2 w_3 f^{(1,0,0)}[t_1, t_2, t_3] + t_1 u_1^2 w_3 f^{(1,0,0)}[t_1, t_2, t_3], \langle\langle 7 \rangle\rangle, \frac{w_3}{t_1 t_2} \right\}$$

**R3r = TGchecks // TG<sub>2,3</sub> // TG<sub>1,3</sub> // TG<sub>1,2</sub>; R3l - R3r**

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

**Short [OC1 = TGchecks // TG<sub>1,2</sub> // TG<sub>1,3</sub>]**

$$\left\{ f[t_1, t_2, t_3] v_1 - \langle\langle 1 \rangle\rangle + \langle\langle 15 \rangle\rangle + t_1 u_1^2 w_3 f^{(1,0,0)}[t_1, t_2, t_3], \langle\langle 7 \rangle\rangle, \frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle} \right\}$$

**OCr = TGchecks // TG<sub>1,3</sub> // TG<sub>1,2</sub>; OC1 - OCr**

$$\{0, -f[t_1, t_2, t_3] u_1 u_2 w_3 + f[t_1, t_2, t_3] t_1 u_1 u_2 w_3 + f[t_1, t_2, t_3] u_1 u_3 w_3 - f[t_1, t_2, t_3] t_1 u_1 u_3 w_3, \\ -f[t_1, t_2, t_3] u_1 u_2 w_2 + f[t_1, t_2, t_3] t_1 u_1 u_2 w_2 + f[t_1, t_2, t_3] u_1 u_3 w_2 - f[t_1, t_2, t_3] t_1 u_1 u_3 w_2, \\ 0, 0, 0, 0, 0, 0\}$$

## A Finite-Rank Turbo-Gassner Representation

$$\eta /: \eta[i\_ ]^2 = 0; \eta /: \eta[i\_ ] \eta[j\_ ] = 0;$$

$$\text{FTG}_{i,j}[f] := \text{Expand}[f /. \{ \\ f\_ \cdot v_{k\_} \Rightarrow \text{Plus}[f v_k /. v_j \rightarrow (1 - t_i - \eta[i]) v_i + (t_i + \eta[i]) v_j, \\ (t_i \text{Coefficient}[f, \eta[i]] - t_j \text{Coefficient}[f, \eta[j]]) \times \\ (1 - t_i^{-1}) (u_k /. u_j \rightarrow (1 - t_i) u_i + t_i u_j) u_i w_j, \\ K\delta_{k,i} (f /. \_ \eta \rightarrow 0) (u_j - u_i) u_i w_j], \\ u_j \rightarrow (1 - t_i) u_i + t_i u_j, \\ w_i \rightarrow w_i + (1 - t_i^{-1}) w_j, w_j \rightarrow t_i^{-1} w_j \}];$$

$$\text{ff} = f_0 + f_1 \eta[1] + f_2 \eta[2] + f_3 \eta[3];$$

$$\text{FTGchecks} = \{\text{ff} v_1, \text{ff} v_2, \text{ff} v_3, u_1, u_2, u_3, w_1, w_2, w_3\};$$

$$\{v_1, v_2\} // \text{FTG}_{1,2} // \text{Column}$$

$$v_1 - u_1^2 w_2 + u_1 u_2 w_2$$

$$v_1 - t_1 v_1 + t_1 v_2 - v_1 \eta[1] + v_2 \eta[1]$$

**Short [R3l = FTGchecks // FTG<sub>1,2</sub> // FTG<sub>1,3</sub> // FTG<sub>2,3</sub>, 10]**

$$\left\{ f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \frac{f_2 t_2 u_1^2 w_2}{t_1} + f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 + f_2 u_1^2 w_3 - \frac{f_2 u_1^2 w_3}{t_1} + f_1 t_1 u_1^2 w_3 - f_2 t_2 u_1^2 w_3 + \frac{f_2 t_2 u_1^2 w_3}{t_1} - \frac{f_3 t_3 u_1^2 w_3}{t_2} + \frac{f_3 t_3 u_1^2 w_3}{t_1 t_2} - f_2 u_1 u_2 w_3 + f_2 t_2 u_1 u_2 w_3 - f_3 t_3 u_1 u_2 w_3 + \frac{f_3 t_3 u_1 u_2 w_3}{t_2} + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll 7 \gg, \frac{w_3}{t_1 t_2} \right\}$$

**R3r = FTGchecks // FTG<sub>2,3</sub> // FTG<sub>1,3</sub> // FTG<sub>1,2</sub>; R3l - R3r**

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

**Short [OC1 = FTGchecks // FTG<sub>1,2</sub> // FTG<sub>1,3</sub>]**

$$\left\{ f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \ll 10 \gg + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll 1 \gg, \ll 48 \gg + \ll 1 \gg, \ll 3 \gg, \ll 1 \gg, \frac{w_2}{t_1}, \frac{w_3}{t_1} \right\}$$

**Short [OCr = FTGchecks // FTG<sub>1,3</sub> // FTG<sub>1,2</sub>]**

$$\left\{ f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \ll 10 \gg + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll 1 \gg, \ll 54 \gg + \ll 1 \gg, \ll 3 \gg, \ll 1 \gg, \frac{w_2}{t_1}, \frac{w_3}{t_1} \right\}$$

**OC1 - OCr**

$$\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3, -f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0\}$$

## The Turbo-Burau Representation

$$\eta /: \eta[i\_ ]^2 = 0; \eta /: \eta[i\_ ] \eta[j\_ ] = 0;$$

**TB<sub>i,j</sub>[ξ<sub>-</sub>] :=**

**Expand[ξ /.**

$$f\_ \cdot v_k \rightarrow \text{Plus}[f v_k / . v_j \rightarrow (1 - t - \eta[i]) v_i + (t + \eta[i]) v_j,$$

$$(t - 1)$$

$$(\text{Coefficient}[f, \eta[i]] - \text{Coefficient}[f, \eta[j]]) (u_k / . u_j \rightarrow (1 - t) u_i + t u_j) u_i w_j,$$

$$K\delta_{k,i} (f / . \_ \eta \rightarrow 0) (u_j - u_i) u_i w_j],$$

$$u_j \rightarrow (1 - t) u_i + t u_j,$$

$$w_i \rightarrow w_i + (1 - t^{-1}) w_j, w_j \rightarrow t^{-1} w_j \}];$$

$$\mathbf{ff} = f_0 + f_1 \eta[1] + f_2 \eta[2] + f_3 \eta[3];$$

$$\mathbf{checks} = \{ff v_1, ff v_2, ff v_3, u_1^2 w_1, u_1^2 w_2, u_1, u_2, u_3, w_1, w_2, w_3\};$$

**Short [R3l = checks // TB<sub>1,2</sub> // TB<sub>1,3</sub> // TB<sub>2,3</sub>, 10]**

$$\left\{ f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + t f_1 u_1^2 w_2 + f_2 u_1^2 w_2 - t f_2 u_1^2 w_2 + f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 + t f_1 u_1^2 w_3 + 2 f_2 u_1^2 w_3 - \frac{f_2 u_1^2 w_3}{t} - t f_2 u_1^2 w_3 - f_3 u_1^2 w_3 + \frac{f_3 u_1^2 w_3}{t} - f_2 u_1 u_2 w_3 + t f_2 u_1 u_2 w_3 + f_3 u_1 u_2 w_3 - t f_3 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll 9 \gg, \frac{w_3}{t^2} \right\}$$

**R3r = checks // TB<sub>2,3</sub> // TB<sub>1,3</sub> // TB<sub>1,2</sub>; R3l - R3r**

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

**Short [OC1 = checks // TB<sub>1,2</sub> // TB<sub>1,3</sub>]**

{ <<20>> + f<sub>2</sub> v<sub>1</sub> η[2] + f<sub>3</sub> v<sub>1</sub> η[3], <<54>> + t f<sub>3</sub> v<sub>2</sub> η[3], <<7>>,  $\frac{\langle\langle 1 \rangle\rangle}{t}$ ,  $\frac{w_3}{t}$  }

**OCr = checks // TB<sub>1,3</sub> // TB<sub>1,2</sub>; OC1 - OCr**

{0, -f<sub>0</sub> u<sub>1</sub> u<sub>2</sub> w<sub>3</sub> + t f<sub>0</sub> u<sub>1</sub> u<sub>2</sub> w<sub>3</sub> + f<sub>0</sub> u<sub>1</sub> u<sub>3</sub> w<sub>3</sub> - t f<sub>0</sub> u<sub>1</sub> u<sub>3</sub> w<sub>3</sub>,  
-f<sub>0</sub> u<sub>1</sub> u<sub>2</sub> w<sub>2</sub> + t f<sub>0</sub> u<sub>1</sub> u<sub>2</sub> w<sub>2</sub> + f<sub>0</sub> u<sub>1</sub> u<sub>3</sub> w<sub>2</sub> - t f<sub>0</sub> u<sub>1</sub> u<sub>3</sub> w<sub>2</sub>, 0, 0, 0, 0, 0, 0, 0, 0}