Kapovitch's class, Mon March 23: Rational Homotopy Theory

March-23-15 11:09 AM

Recall: $\exists$ functor $X \to A_{PL}(X)$ (contravariant)

$$\{\text{spaces \& maps}\} \longrightarrow \left\{\begin{matrix}\text{Commutative}\\ \text{DGA}/\mathbb{Q}\end{matrix}\right\}$$

Thm A_{PL} sends rational equivalences to q.i.:

$X \xrightarrow{f} Y$ rational equiv. $\Rightarrow A_{PL}(f): A_{PL}(Y) \to A_{PL}(X)$ is a q.i.

There is also an "inverse"

$$\left\{\begin{matrix}\text{CDGA, connected}\\ \text{simply connected}\\ \text{finite type}\end{matrix}\right\} \longrightarrow \text{Spaces.}$$

We want to classify CDGA's up to q.i.

Def'n A Sullivan Algebra is a CDGA $(\Lambda V, d)$ where V is a graded v.s. s.t. V has a filtration

$$V(0) \subset V(1) \subset V(2) \dots$$

s.t. $d: V(K) \longrightarrow \Lambda V(K-1)$

Thm Every connected CDGA has a Sullivan model. (up to q.i.)

Ex: $\left(\Lambda(x_4, y_7),\ d: \begin{matrix} x_4 \to 0 \\ y_7 \to x_4^2\end{matrix}\right)$ is a Sullivan algebra with $V(0) = \langle 1, x_4 \rangle$

$V(1) = \langle 1, x_4, y_7 \rangle$

Example $\left(\Lambda(x_2, y_3), \begin{array}{l} dx_2 = y_3 \\ dy_3 = 0 \end{array}\right)$ no cohomology, so q.i. to $V = \emptyset$.

Def A minimal Sullivan algebra is one that does not have contractible factors as above. precisely, this is

$$\operatorname{im}(d) \subset \Lambda^+ V \cdot \Lambda^+ V$$

Thm Every connected CDGA A has a minimal model M_A.

Example

$$\Lambda(x_2, x_4, y_3, y_7), \quad \begin{array}{l} dx_2 = dx_4 = 0 \\ dy_3 = x_4 - x_2^2 \\ dy_7 = x_4^2 \end{array}$$

is equiv to

$$\Lambda(a_2, b_7) \quad da_2 = 0 \quad db_7 = a_2^4$$

via $a_2 \longrightarrow x_2$, $b_7 \longrightarrow y_7 - y_3 x_4$ (?)

w/ q-inverse ?

we also have that if $f: A \longrightarrow B$ is q.i. then $\exists \hat{f}: M_A \longrightarrow M_B$ which is an isomorphism.

Thm If $A = A_{PL}(X)$ & $M_A = (\Lambda V, d)$

Then $V_i \cong \mathrm{Hom}_{\mathbb{Q}}(\pi_i(X) \otimes \mathbb{Q}, \mathbb{Q})$

Ex S^{2n+1}, $H^i = 0, 2n+1$, so

$$M_A = (\Lambda y_{2n+1},\ dy_{2n+1} = 0)$$

Ex S^{2n} $\quad A = H^*(S^{2n}) = 0, 2n$

$a_{2n} \quad da_{2n} = 0$

subject to $a_{2n}^2 = 0$ {Not free!}

$$M_A = \left(\Lambda(x_{2n}, y_{4n-1}) : \begin{array}{l} dx_{2n} = 0 \\ dy_{4n-1} = x_{2n}^2 \end{array}\right)$$

q.i. to A via $\quad x \longrightarrow a$

$\quad y \longrightarrow 0$

so $\pi_i(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 0, 2n, 4n-1 \\ 0 & \text{otherwise} \end{cases}$