Georgetown-1503 handout on Feb 20

February-20-15 7:05 AM

45 minutes talk!

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ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Georgetown-1503

When does a group have a Taylor expansion?

Abstract. It is insufficiently well known that the good old Taylor expansion has a completely algebraic characterization, which generalizes to arbitrary groups (and even far beyond). Thus one may ask: Does the braid group have a Taylor expansion? (Yes, using iterated integrals and/or associators). Do braids on a torus ("elliptic braids") have Taylor expansions? (Yes, using more sophisticated iterated integrals / associators). Do virtual braids have Taylor expansions? (No, yet for nearby objects the deep answer is Probably Yes). Do groups of flying rings (braid groups one dimension up) have Taylor expansions? (Yes, easily, yet the link to TQFT is yet to be fully explored).

Brook Taylor

Expansions for Groups. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i \colon a_i \in \mathbb{Q},\ g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i \colon \sum a_i = 0\} \subset \mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A} = \operatorname{gr}\mathcal{K} := \bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

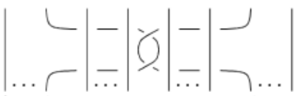
P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, a "multiplicative expansion" if in addition it preserves the product, and a "Taylor expansion" if it also preserves the co-product, induced from the diagonal map $G \to G\times G$.

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f\colon f(0)=0\}$. Then $\mathcal{I}^m = \{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is the unique Taylor expansion!

Pure Braids. Take $G = PB_n = \pi_1(C_n = \mathbb{C}^n \setminus \text{diags})$. It is generated by the love-behind-the-bars braids σ_{ij}, modulo "Reidemeister moves". $\mathcal{I}$ is generated by $\{\sigma_{ij}-1\}$ and $\mathcal{A}$ by $\{t_{ij}\}$, the classes of the $\sigma_{ij}-1$ in $\mathcal{A}_1 = \mathcal{I}/\mathcal{I}^2$. Reidemeister becomes



$$[t_{ij}+t_{ik}, t_{jk}] = 0 \quad \text{and} \quad [t_{ij}, t_{kl}] = 0.$$

Theorem. For $\gamma\colon [0,1] \to C_n$, with z_i its ith coordinate, the iterated integral formula on the right defines a Taylor expansion for PB_n.

$$Z(\gamma) = \sum_{\substack{m\geq 0\\ 0<t_1<\dots<t_m<1\\ 1\leq i_1<j_1, i_2<j_2,\dots,i_m<j_m\leq n}} \prod_{\alpha=1}^{m} \frac{t_{i_\alpha j_\alpha}}{2\pi i} d\log(z_{i_\alpha} - z_{j_\alpha}),$$

Commments. • I don't know a combinatorial/algebraic proof that PB_n has a Taylor expansion. • I don't know if every "partial expansion" extends. • This is the seed for the Drinfeld theory of associators!

Picture.

Elliptic Braids.

Pictures of Calaque, Enriquez, ...

Virtual Braids.

Peter Lee

Flying Rings.

Pictures of Malcev & Quillen

Comment: Generalizes to arbitrary alg. structures.

picture for formula, picture of Kohno

References.

Elliptic Braids:

Pictures of Calaque, Enriquez, Etingof

shamelessly steal formulas from Enriquez.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Add something about GT/GRT?

Enriquez formulas in Feb 15 page.