Georgetown-1503 handout on Feb 15

February-15-15 11:04 AM

45 minutes talk!

Add a "computations" back page!

Add a major disclaimer.... ✓

Video, links, and more @ Dror Bar-Natan: Talks: Georgetown-1503:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Georgetown-1503

When does a group have a Taylor expansion?

Brook Taylor

Abstract. It is insufficiently well known that the good old Taylor expansion has a completely algebraic characterization, which generalizes to arbitrary groups (and even far beyond). Thus one may ask: Does the braid group have a Taylor expansion? (Yes, using iterated integrals and/or associators). Do braids on a torus ("elliptic braids") have Taylor expansions? (Yes, using more sophisticated iterated integrals / associators). Do virtual braids have Taylor expansions? (No, yet for nearby objects the deep answer is Probably Yes). Do groups of flying rings (braid groups one dimension up) have Taylor expansions? (Yes, easily, yet the link to TQFT is yet to be fully explored).

The Taylor Expansion.

Expansions for Groups.

Pure Braids.

Elliptic Braids.

put the PB_n presentation & add a confession

shamelessly steal formulas from Enriquez.

Virtual Braids.

Flying Rings.

Also put KZ & associators.

Add something about GT/GRT?

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Elliptic braids formulas.

From Enriquez "Elliptic Associators":

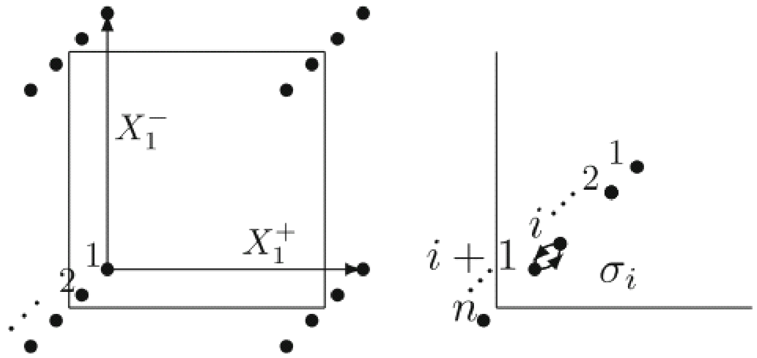

Fig. 1 Generators of the elliptic braid group $B_{1,n}$

The group $B_{1,n}(n \geq 1)$ can be presented by generators $\sigma_i(i = 1, \dots, n-1)$, $X_1^\pm$, and relations

A presentation of the elliptic braid group.

$$
\begin{gathered}
(\sigma_1^{\pm 1} X_1^\pm)^2 = (X_1^\pm \sigma_1^{\pm 1})^2, \quad (X_1^\pm, \sigma_i) = 1 \quad \text{for } i = 2, \dots, n-1, \\
(X_1^-, (X_2^+)^{-1}) = \sigma_1^2, \quad X_1^\pm \cdots X_n^\pm = 1, \quad (\sigma_i, \sigma_j) = 1 \quad \text{for } |i-j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \dots, n-2,
\end{gathered}
\tag{7}
$$

Birman's book

where $X_{i+1}^\pm = \sigma_i^{\pm 1} X_i^\pm \sigma_i^{\pm 1}$ for $i = 1, \dots, n-1$ (see [4] and Fig. 1). In particular, $P_{1,1} = B_{1,1} = \{1\}$, and $P_{1,2}$ is the free group with two generators $X_1^\pm$.

We also define $\mathfrak{t}_{1,S}^{\mathbf{k}}$ as the $\mathbf{k}$-Lie algebra with generators $x_i^\pm$, $i \in S$ and relations $\sum_{i \in S} x_i^\pm = 0$, $[x_i^\pm, x_j^\pm] = 0$ for $i \neq j$, $[x_i^+, x_j^-] = [x_j^+, x_i^-]$ for $i \neq j$, $[x_k^\pm, [x_i^+, x_j^-]] = 0$ for i, j, k distinct. We then have a Lie algebra morphism $\mathfrak{t}_S^{\mathbf{k}} \to \mathfrak{t}_{1,S}^{\mathbf{k}}$, $t_{ij} \mapsto [x_i^+, x_j^-]$, which we denote by $x \mapsto \{x\}$. We will also write $t_{ij} = [x_i^+, x_j^-]$. We define $\hat{\mathfrak{t}}_{1,S}^{\mathbf{k}}$ as the degree completion of $\mathfrak{t}_{1,S}^{\mathbf{k}}$, where $\deg(x_i^\pm) = 1$.

Definition 4.1 The set $\underline{Ell}(\mathbf{k})$ of elliptic associators defined over $\mathbf{k}$ is the set of quadruples (μ, Φ, A_+, A_-), where $(\mu, \Phi) \in \underline{M}(\mathbf{k})$ and $A_\pm \in \exp(\hat{\mathfrak{t}}_{1,2}^{\mathbf{k}})$, such that:

$$
\alpha_\pm^{3,1,2} \alpha_\pm^{2,3,1} \alpha_\pm^{1,2,3} = 1, \text{ where } \alpha_\pm = \{e^{\pm\mu(t_{12}+t_{13})/2}\} A_\pm^{1,23} \{\Phi^{1,2,3}\}, \tag{25}
$$

$$
\begin{aligned}
\{e^{\mu t_{12}}\} = \big(&\{\Phi\}^{-1} A_-^{1,23} \{\Phi\}, \{e^{-\mu t_{12}/2} (\Phi^{2,1,3})^{-1}\} (A_+^{2,13})^{-1} \\
&\{\Phi^{2,1,3} e^{-\mu t_{12}/2}\}\big).
\end{aligned}
\tag{26}
$$

Proposition 4.8 *There is a unique scheme morphism* $\sigma : \underline{M} \to \underline{Ell}$, $(\mu, \Phi) \to (\mu, \Phi, A_+, A_-)$, *where*

$$
A_+ := \Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right) \cdot e^{\mu \frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2)} \cdot \Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right)^{-1},
$$

$$
A_- := e^{\mu t_{12}/2} \Phi\left(\frac{\operatorname{ad} x_2}{e^{\operatorname{ad} x_2} - 1}(y_1), t_{21}\right) e^{x_1} \Phi\left(\frac{\operatorname{ad} x_1}{e^{\operatorname{ad} x_1} - 1}(y_2), t_{12}\right)^{-1}
$$

(we set $x_i := x_i^+$, $y_i := x_i^-$). It is compatible with the semigroup scheme morphism $\underline{\mathrm{GT}}(-) \to \underline{\mathrm{GT}}_{ell}(-)$ from Proposition 3.21.

Then there is a whole chapter about KZB ...