Crooks: Equivariant Cohomology and the Localization Theorem

March-11-15 9:09 AM

① context: Fix a commutative ring k (a base ring)

Recall ordinary singular cohomology is a contravariant functor

$$H^*: \text{Top spaces} \longrightarrow \begin{matrix}\mathbb{Z}\text{-graded comm.}\\ k\text{-algebras}\end{matrix}$$

Fix a topological group G. We will construct an analogous functor on G-spaces. More precisely,

cont. left action

$$H^*_G: \left\{\begin{matrix}G\text{-}\\ \text{spaces}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-graded comm.}\\ k\text{-algebras}\end{matrix}\right\}$$

Naive attempt:

$${}^N H^*_G := H^*(X/G)$$

will give the same answer whenever G acts transitively on X, no matter the size of the orbit.

Actual def: $\exists$ contractible space EG on which G acts freely. So

$$EG \longrightarrow EG/G =: BG$$

is a principal G-bundle "the universal principal G-bundle".

<u>Def</u>

$$H^*_G(X) := H^*(X_G)\text{, where}$$

$$\left(X_G = EG \times X/G\right) \longrightarrow BG$$

"the Borel mixing space". (well-defined!)

a functor!

Properties:

(1) $H^*_G(pt) = H^*(BG)$

(2) $(X \to pt) \Rightarrow H^*_G(pt) \longrightarrow H^*_G(X)$

So $H^*_G(X)$ is a module over $H^*_G(pt)$ (functorially)

(3) If $G \circlearrowright X$ trivially, then

$$H^*_G(X) = H^*(X) \otimes H^*_G(pt)$$

$K = \mathbb{C}$ here, G - compact connected Lie.

(4) If $G \circlearrowright X$ w/ finite stabilizers,

$$H^*_G(X) \cong H^*(X/G)$$

(5) $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ a G-equivariant complex vector bundle,

get a vector bundle $\begin{array}{c} V_G \\ \downarrow \end{array}$

get a vector bundle $\begin{array}{c} V_G \\ \downarrow \\ X_G \end{array}$

set $C_n^G(V) = \underset{\substack{\uparrow \\ \text{chern class}}}{C_n}(V_G) \in H^*(X_G) = H_G^*(X)$

Example, S^1: $S^1 \hookrightarrow S^{2n-1} \subset \mathbb{C}^n$

$S^1 \subset S^3 \subset S^5 \subset S^7$ a sequence of S^1-equivariant inclusions

So S^1 acts on $\varinjlim S^{2n+1} = S^\infty$ which is contractible.

so $ES^1 = S^\infty$ $BS^1 = ES^1/S^1 = \mathbb{C}P^\infty$

so
$H_{S^1}^*(pt) = H^*(\mathbb{C}P^\infty) = \mathbb{C}[t]$, $\deg t = 2$.

Example $G = T$ is a torus of rank r,

$X^*(T) = \mathrm{Hom}(T, S^1)$ (group maps)

$\subset (\mathfrak{t}_\mathbb{C})^*$

The Lie algebra of T.

$H_T^*(pt) = \mathrm{Sym}((\mathfrak{t}_\mathbb{C})^*)$ by $\alpha \in X^*(T)$ goes to $C_1^T(\mathbb{C}_\alpha)$

Example $K \subseteq G$ a closed subgroup:

$$H^*_G(G/K) = H^*\left(\frac{EG\times(G/K)}{G}\right) = H^*(EG/K)$$

but as a K-space, $EK = EG$, so

$$= H^*(EK/K) = H^*(BK)$$

<u>Prop</u> $T\subset G$ a maximal torus. Then

a. $H^*_G(X) \cong H^*_T(X)^W$

b. $H^*_T(X) \cong H^*_T(pt) \otimes_{H^*_T(pt)^W} H^*_G(X)$.

⋮

<u>Example</u> $T\subset G$ a maximal torus, G/T is a flag variety

$$H^*_T(G/T) = H^*_T(pt)\otimes_{H^*_T(pt)} H^*_G(G/T)$$

$$= H^*_T(pt)\otimes_{H^*_T(pt)^W} H^*_T(pt)$$

$$= (\text{poly})\otimes_{\text{sym poly}}(\text{Poly}).$$

Up to here this is all Borel's Thesis!

The Localization Theorem

* T a compact torus.
* X a smooth complex projective $T_{\mathbb{C}}$-variety.

$$T_{\mathbb{C}} := (\mathbb{C}^*)^r \quad \text{if } T = (S^1)^r$$

We have a T-equivariant inclusion

$X^T \hookrightarrow X$, hence

$$i^*: H_T^*(X) \longrightarrow H_T^*(X^T) \quad \text{"restriction"}$$

i^* is a key to understanding $H_T^*(X)$

<u>Thm</u> i^* is injective.

<u>Question</u> is i^* also surjective?

<u>Note</u> $(X, X - X^T) \longrightarrow$ long exact seq:

$$\to H_T^j(X, X-X^T) \longrightarrow H_T^j(X) \longrightarrow H_T^j(X - X^T)$$

H^{j+1}

only morally.

By Thom, $H_T^j(X, X-X^T) \cong H_T^{j-d}(X^T)$

So morally,

$$\to H_T^{j-d}(X^T) \xrightarrow{i_*} H_T^j(X) \longrightarrow H_T^j(X - X^T)$$

a map going the other way!

a map going the other way!

<u>Thm</u> $i^* \circ i_* : H_T^*(X^T) \to H_T^*(X^T)$

is multiplication by a fixed element

$e_T(N) \in H_T^*(X^T)$

N: The normal bundle of X^T in X

e_T: The T-equivariant Euler class

$= C_{top}^T$.

<u>problem</u> mult. by $e_T(N)$ is not surjective in general.

<u>Sol'n</u> Localize $H_T^*(X)$ & $H_T^*(X^T)$ as modules over $H_T^*(pt)$ so that mult. by $e_T(N)$ becomes surjective.

$\rightsquigarrow$ Leads to the localization thm.

<u>Thm</u> (Localization) There exist finitely many $\alpha_1, \dots \alpha_n \in X^*(T) \subseteq H_T^*(pt)$ s.t. i^* is an isomorphism after $\alpha_1, \dots \alpha_n$ are inverted.

<u>Sketch of pf</u> (when X^T is finite)

$X^T = \{x_1, \dots x_r\}$ $N = \{T_{x_1}X \quad T_{x_2}X \quad ?$

$X^T = \{x_1, \dots x_k\}$ $\quad N = \left\{ \begin{matrix} T_{x_1}X & & T_{x_2}X & \\ \downarrow & & \downarrow & \dots \\ x_1 & & x_2 & \end{matrix} \right\}$

$T_{x_j}X \cong \bigoplus_{l=1}^{m} \mathbb{C}_{\alpha_{l,j}}$ as reps of T.

$H_T^*(X^T) = \bigoplus_{j=1}^{k} H_T^*(pt)$ $\qquad e_T(N) \longrightarrow (\overbrace{\dots\dots}^{k\text{-tuple}})$

$e_T(N) = (\dots \underbrace{\prod_{l=1}^{m} \alpha_{l,j}}_{\text{in pos } j} \dots)$