Yampolsky's Class, Wed Jan 21

January-21-15 1:05 PM

Reminder on extremal length:

Γ: a family of loc. rectifiable curves in $\mathbb{C}$.

$\rho: \mathbb{C} \to \mathbb{R}$ is "admissable" if $\rho \geq 0$ &

$$A(\rho) = \iint_{\mathbb{C}} \rho^2 \, dx\,dy \neq 0, \infty$$

$L_\gamma(\rho) = \int_\gamma \rho |dz|$ (may be ∞)

$L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho) \qquad \lambda(\Gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}$

Theorem Let f be K-qc, $\Gamma' = f(\Gamma)$. Then

$$K^{-1}\lambda(\Gamma) \leq \lambda(\Gamma') \leq K\lambda(\Gamma)$$

Corollary λ is conformally invariant.

Proof of Thm Set $\rho' = \dfrac{\rho}{|f_z| - |f_{\bar z}|} \circ f^{-1}$ in

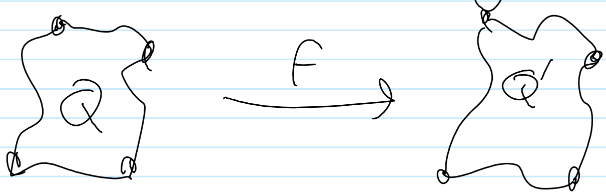
$\Omega' = f(\Omega)$. Then $\left(d\zeta = f_z dz + f_{\bar z} d\bar z \right)$

$$\int_{\gamma' \in \Gamma'} \rho' |d\zeta| \geq \int_\gamma \rho |dz|$$

Now $\displaystyle \iint \rho'^2 \, d\xi\, d\eta = \iint \rho^2 \underbrace{\frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}}_{\leq K} dx\,dy$ TBC....

Def'n: $f: \Omega \to \mathbb{C}$ is K-qc (geometric def'n) if * f is homeo+

* Let $Q \subset \Omega$ be any quadrilateral



Then $\operatorname{mod}(Q') \le K \operatorname{mod}(Q)$

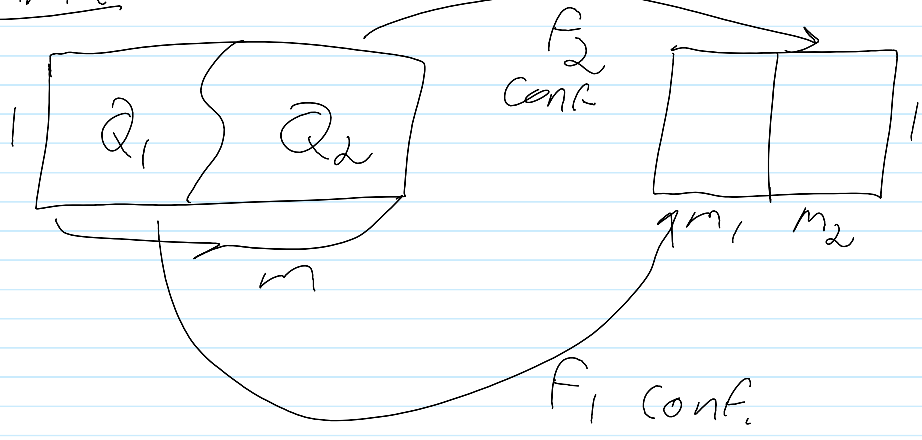
(This implies $K^{-1}\operatorname{mod}(Q) \le \operatorname{mod}(Q')$,
by changing the role of horizontal/vertical
in Q/Q')

<u>Claim</u> Analytic K-qC $\Rightarrow$ geometric K-qC
(follows from the first half of Thm)

To prove Geom $\Rightarrow$ Anal, we need to show
ACL . . .

Weyl's Lemma: 1-qC maps are conformal

<u>Lemma^2</u>:



then $m = m_1 + m_2$ iff the dividing line
is vertical.

<u>Pf of Lemma^2</u> . . . .

Pf of Lemma 2 ....

Pf of 1-QC is conformal: ....

Mori's Theorem $f: \mathbb{D} \to \mathbb{D}$ K-QC,

$f(0)=0$, Then

$$|f(z_1)-f(z_2)| \leq 16\, |z_1 - z_2|^{1/K}$$ (Proof later)

16 is sharp as a uniform (over K) bound!

Corollaries

1. K-QC maps extend continuously to the bndry S^1.
2. Normalized K-QC maps $\mathbb{D} \to \mathbb{D}$ form an equi-continuous function.
3. Normalized K-QC maps $\mathbb{D} \circlearrowleft$ are pre-compact in the compact-open sense.

Extremal problems

1. (Grötzsch) Let R_1 be an annulus which separates $\mathbb{D}$ from a pt. R, $R>1$. Then $*$ sup Mod(R_1) is achieved on

$$\mathbb{C} \setminus (\mathbb{D} \cup [R, \infty))$$

$* \quad \sup \mathrm{Mod}(R_1) = \frac{1}{2\pi} 4R$

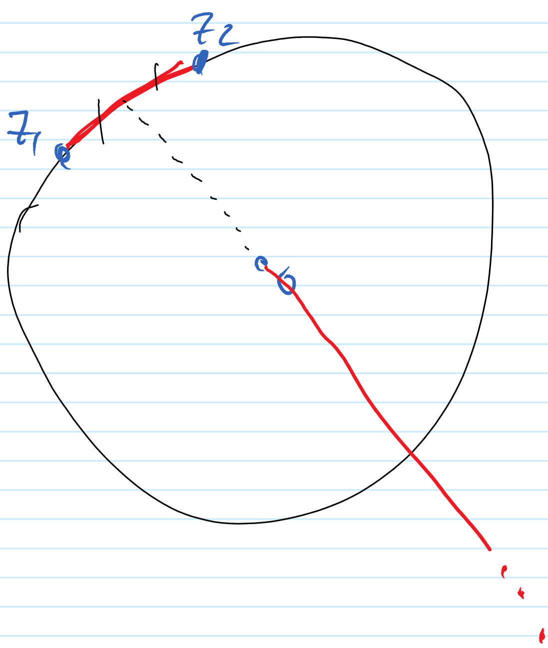
2. (Teichmüller) Let R_2 be an annulus that separates $\{-1, 0\}$ from $p > 0$, then the extremal R_2 is

$$\mathbb{C} \setminus \left([-1, 0] \cup [p, \infty)\right)$$

and

$$\sup \mathrm{Mod}(R_2) = \ldots$$

3. (Mori) $z_1, z_2 \in S^1$, $|z_1 - z_1| > \lambda$, R_3 separates z_1, z_2 from 0. Then the extremal R_3 is



and

$$\sup \mathrm{mod}(R_3) = \frac{1}{2\pi} \log \frac{16}{\lambda}$$