Yampolsky's Class, Mon Jan 12

January-12-15 1:10 PM

S a Riemann surface

$$\begin{array}{c} M \\ \downarrow p \\ S \end{array} \quad \text{univ cover}$$

$M = \hat{\mathbb{C}}$ or $\mathbb{C}$ or $\mathbb{D}$
(i) (ii) (iii)

$S = M/\Gamma$, $\Gamma = \pi_1(S)$

Γ acts freely & properly discontinuously on M.

Case (i). Γ must be $\{e\}$.

Case (ii). $\Gamma = \{e\}$ or $\mathbb{Z}$ or $\mathbb{Z}^2$

$S = \hat{\mathbb{C}}$ or $\mathbb{C}^*$ or $\mathbb{T}_\Gamma$

In the last case Γ is a lattice Λ so we denote $\mathbb{T}_\Lambda$.

Case (iii) $S = \mathbb{D}/\Gamma = \mathbb{H}/\Gamma$

$$\mathrm{Aut}(\mathbb{H}) = PSL_2(\mathbb{R})$$

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{z-a}{1-\bar{a}z} \right\}$$

$\exists!$ invariant metric on $\mathbb{H}$, the hyperbolic metric.

on $\mathbb{D}$: $\dfrac{2|dz|}{1-|z|^2}$ on $\mathbb{H}$: $ds = \dfrac{|dz|}{\mathrm{Im}\, z}$

So there is a hyperbolic metric on S.

Useful classification of automorphisms of $\mathbb{D}$:

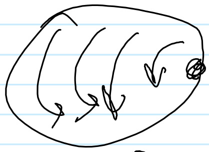
$\gamma \in \mathrm{Aut}(\mathbb{D})$

Case I: γ has a f.p. in $\mathbb{D}$. It is a unique f.p. in $\overline{\mathbb{D}}$. γ is conjugate to a rotation.

"elliptic". $\gamma \neq e$ cannot be in Γ!

Case II: γ has a double f.p. on $\partial\mathbb{D} = S^1$.

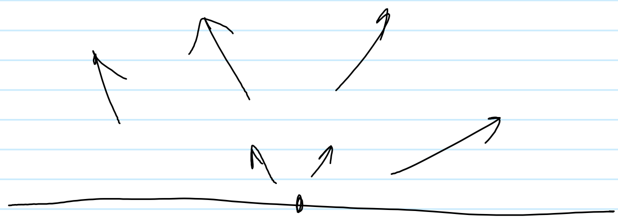
"parabolic": Translation on $\mathbb{H}$;



on $\mathbb{D}$

Case III: γ has two f.p. on S^1

"hyperbolic". γ is conjugate to a scaling by a positive constant.



In general, there is a geodesic connecting the two fixed point, "The axis of γ"

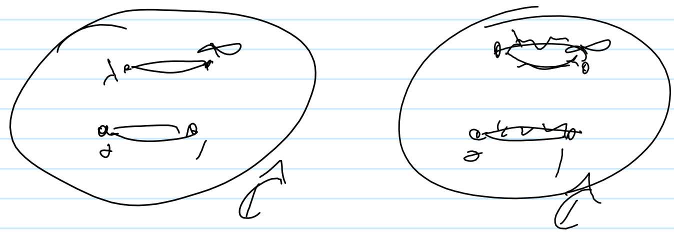
Pick $\lambda \notin \{0, 1, \infty\}$ & consider the elliptic curve

$$\underset{\mathbb{C}^2}{\underset{\cap}{R}} = \{(z,w) : w^2 = z(z-1)(z-\lambda)\} \xrightarrow[\pi]{z} \mathbb{C}$$

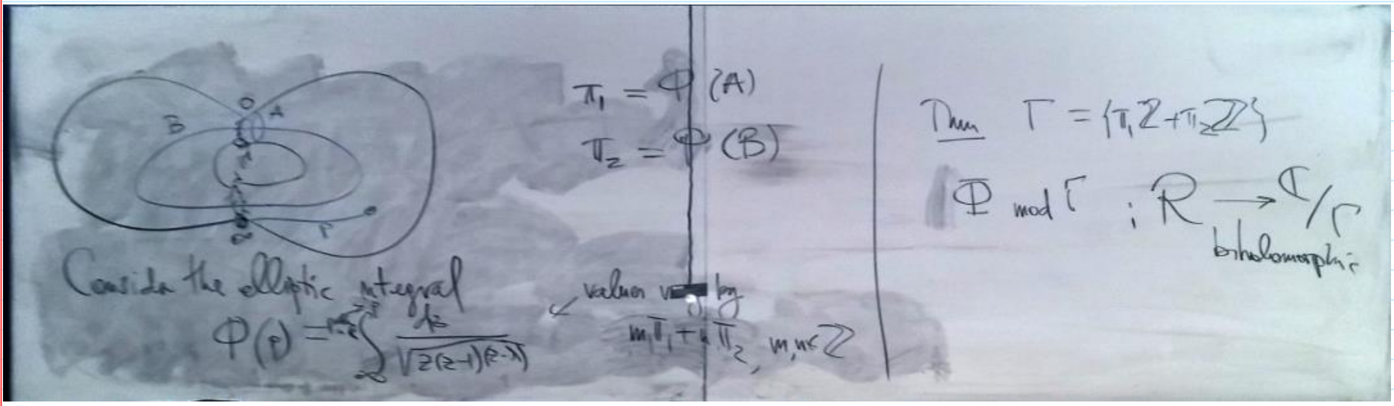
Has a complex structure by requiring that z is holomorphic.

R is a torus!

R is the Riemann surface of $\sqrt{z(z-1)(z-\lambda)}$



Glue as edges.

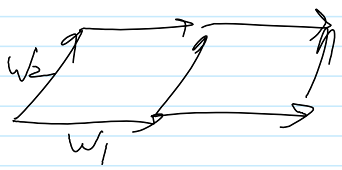


$\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$

Def'n The conformal moduli space of $\mathbb{T}$:

$$[\mathbb{T}_\Lambda] = \{\mathbb{T}_{\Lambda'} : \exists \text{ conformal iso } \mathbb{T}_\Lambda \to \mathbb{T}_{\Lambda'}\}$$

Def'n A marked $\mathbb{T}_\Lambda$ is



$(\mathbb{T}_\Lambda, w_1, w_2)$

(so that a torus w/ a basis of its π_1 / H_1)

Def'n Teichmüller of $\mathbb{T}$ is the set of triples $(\mathbb{T}_\Lambda, w_1, w_2)$ where two such triples are considered equivalent if there is a conformal $\varphi : \mathbb{T}_\Lambda \to \mathbb{T}_{\Lambda'}$ s.t.

$$\varphi^* : w_i \mapsto w_i' .$$

$$\hat{\mathbb{C}} \xrightarrow{H} \hat{\mathbb{C}}$$

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "H"] \arrow[d] & \hat{\mathbb{C}} \arrow[d] \\
\mathbb{T}_{\Lambda_1} \quad \omega_1, \omega_2 \arrow[r, "\text{conf}", "h"'] & \eta_1, \eta_2 \quad \mathbb{T}_{\Lambda_2}
\end{tikzcd}$$

w.log $H(0)=0$ so $H(z)=az$. Also

$H(\omega_i)=\eta_i$ so $\eta_i = a\omega_i$ so

marked tori are equiv iff

$$\frac{\eta_1}{\eta_2} = \frac{\omega_1}{\omega_2} =: \tau \qquad \left(\text{wlog } \Lambda \text{ is } \; \tau \nearrow, \; \longrightarrow 1\right)$$

wlog, $\tau \in \mathbb{H}$ (it is not real, & $\tau \leftrightarrow \tau^{-1}$)

So Teichmüller space here $= \mathbb{H}$

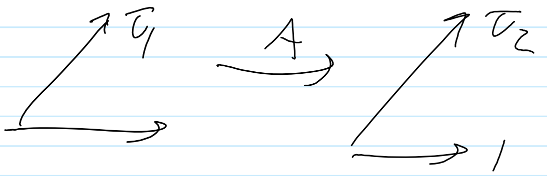
<u>Def'n</u> Teichmüller metric on $\mathbb{H} = \mathrm{Teich}(\mathbb{T})$

$$\mathrm{dist}_T([S_1],[S_2]) = \inf_{\substack{h \in \mathrm{Diff}^+ \\ h:[S_1] \mapsto [S_2]}} \log K_h$$

two marked tori

<u>Thm</u>

in the definition above, $\exists!$ h realizing the infimum, & h is affine — it is A.



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

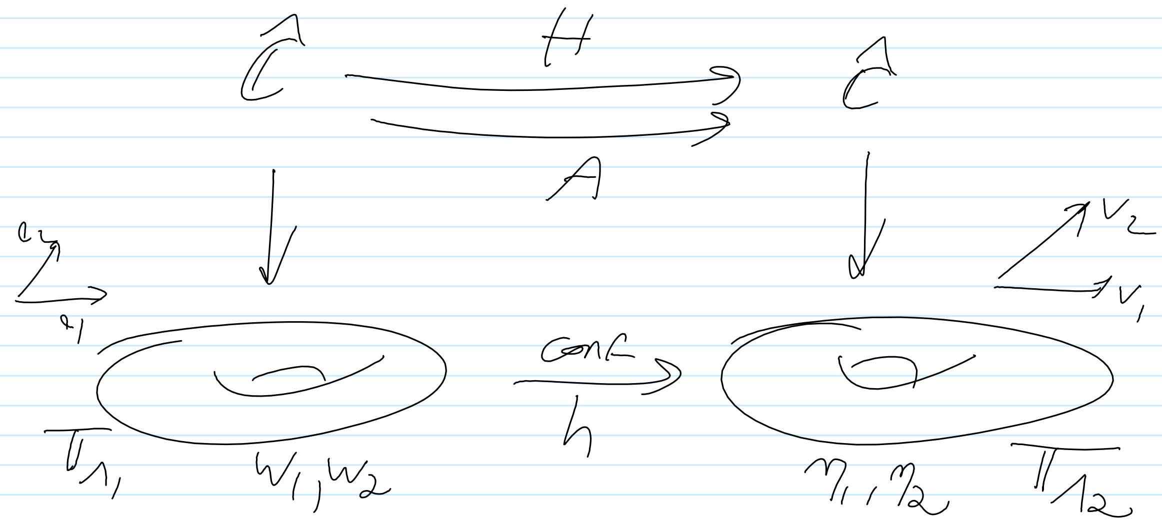
<u>Birkhoff</u> <u>Ergodic</u> <u>Theorem</u>: Suppose (X, μ) is a probability space & $f^t : X \circlearrowleft$ is flow on X preserving μ. Let $\lambda : X \to \mathbb{R}$ be integrable. Then for almost all $x \in X$ the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \lambda(f^t x)\, dt =: \bar{\lambda}(x)$$

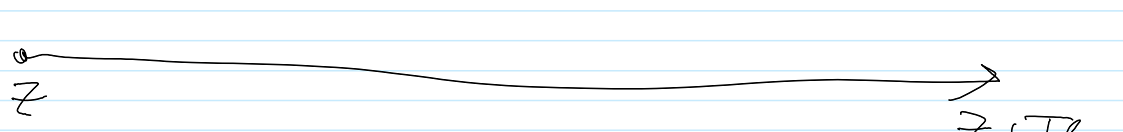
exists, $\int \lambda \, d\mu = \int \bar{\lambda} \, d\mu$

[and $\bar{\lambda}$ is the projection of λ on invariant functions]

<u>pf</u> <u>of</u> <u>thm</u>:



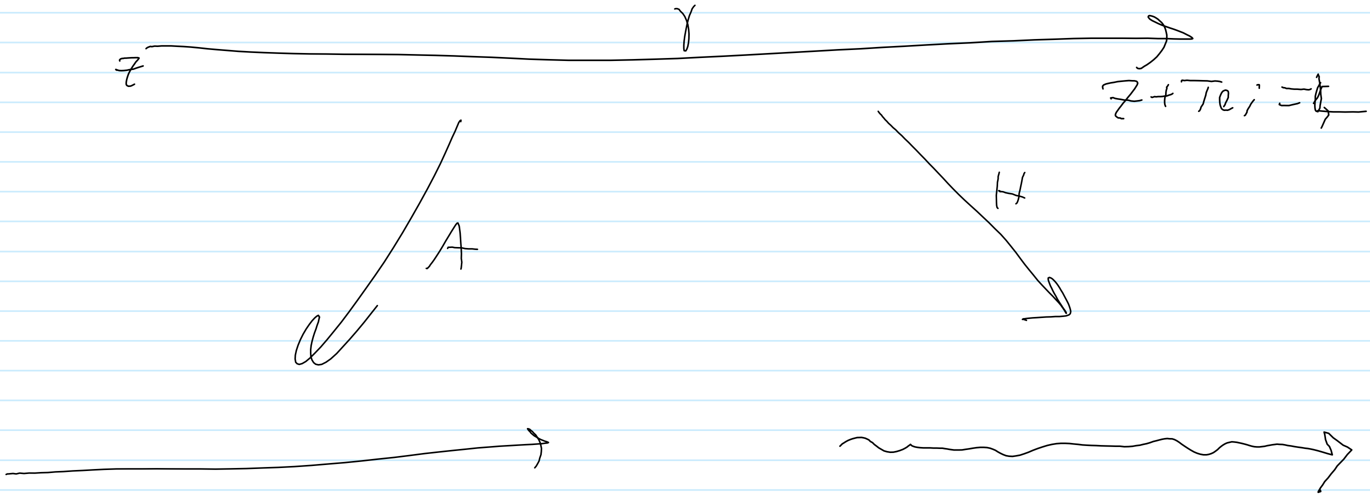
In $\mathbb{C}$, consider the long horizontal line



$T \gg 0$

$$\left| H(z+Te_1) - A(z+Te_1) \right|$$ is unif. bounded in T.

(in fact, $|H(z) - A(z)|$ is bounded)



$$|H(\gamma_T)| \geq |A(\gamma_T)| - C$$

$$\int_{\gamma_T} |\partial_\zeta H| \geq |A(\gamma_T)| - C$$