Some Group Completions

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On the Associated Graded Ring of a Group Ring

DANIEL G. QUILLEN*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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In this paper we show that if the group ring KG of a group G over a field K is filtered by the powers of its augmentation ideal, then the associated graded ring is isomorphic to the universal enveloping algebra of the p-Lie algebra $\mathrm{gr}^p_\cdot G \otimes_Z K$, where $\mathrm{gr}^p_\cdot G$ is the graded p-Lie algebra associated to the p-lower central series of G and where p is the characteristic exponent of K. The proof uses ideas of Lazard's thesis [*1*].

From Brochier's http://www.impan.pl/~burgunde/WSBC09/Drinfeld_Associator_Brochier.pdf:

3. PRO-UNIPOTENT COMPLETION OF A GROUP OF FINITE TYPE

Let G be a group generated by $\{g_1, \dots, g_n\}$ and relations $\{R_1, \dots, R_p\}$. Let $\mathfrak{a}$ be the quotient of the complete free k-Lie algebra on generators $\{\gamma_1, \dots, \gamma_n\}$ by relations

$$\log R_i(e^{\gamma_1}, \dots, e^{\gamma_n}) = 0, \ \forall i = 1 \dots p$$

Denote by $G(k)$ the Lie group of $\mathfrak{a}$, i.e. $G(k) = \exp(\mathfrak{a}) = \{e^a, \ a \in \mathfrak{a}\}$. It is called the *$k$-pro-unipotent completion* of G.

From Hain's "Higher Albanese Manifolds", http://www.cecm.sfu.ca/~nbruin/banff2007/hain.pdf:

1. Unipotent completion

Let Γ be a discrete group. Let k be a field of characteristic 0. A *unipotent group* is a closed subgroup of the subgroup of $\mathrm{GL}_n(k)$ consisting of upper triangular matrices with 1s on the diagonal. Unipotent groups correspond to nilpotent Lie algebras via the logarithm and exponential maps, which are polynomial bijections.

Define the pro-unipotent group

$$\Gamma^{\mathrm{un}}_{/k} := \varprojlim_{\substack{\rho\colon \Gamma \to U(k) \\ \text{Zariski dense} \\ U \text{ unipotent}}} U.$$

It is also π_1 of the Tannakian category of unipotent representations of Γ over k.

Define the pro-nilpotent Lie algebra

$$\mathrm{Lie}(\Gamma^{\mathrm{un}}_{/k}) := \varprojlim \mathrm{Lie}(U).$$

A homomorphism $\Gamma \to U(k)$ from Γ to the k-points of a unipotent k-group U induces a homomorphism $\Gamma^{\mathrm{un}}_{/k} \to U$. The original representation factors $\Gamma \to U(k) \to \Gamma^{\mathrm{un}}_{/k} \to \Gamma^{\mathrm{un}}(k)$.

Let J be the kernel of the augmentation map $k\Gamma \xrightarrow{\epsilon} k$ sending each $\gamma \in \Gamma$ to 1. Define

$$(k\Gamma)^\wedge := \varprojlim_r (k\Gamma/J^r).$$

Then

$$\Gamma^{\mathrm{un}}_{/k} = \{x \in (k\Gamma)^\wedge : \Delta x = x \otimes x\} - \{0\}.$$

From Vezzani's "The Pro-Unipotent Completion", http://perso.univ-rennes1.fr/alberto.vezzani/Files/Research/prounipotent.pdf:

Definition 1. Given an abstract group Γ [resp. a Lie algebra $\mathfrak{g}$], the pro-unipotent completion Γ^{un} [resp. $\mathfrak{g}^{un}$] is the universal pro-unipotent algebraic group G endowed with a map $\Gamma \to G(\mathbb{Q})$ [resp. $\mathfrak{g} \to \mathrm{Lie}(G)$].

Let us focus on the case of groups. The meaning of the definition is that there is a map $u\colon \Gamma \to \Gamma^{un}(\mathbb{Q})$ such that for any map $f\colon \Gamma \to G(\mathbb{Q})$ to the $\mathbb{Q}$-points of a pro-unipotent algebraic group G, there exists a unique map $\phi\colon \Gamma^{un} \to G$ such that $f = \phi(\mathbb{Q}) \circ u$. In other words, we are looking for a left adjoint to the functor $G \mapsto G(\mathbb{Q})$ defined from pro-unipotent algebraic groups to abstract groups. Sadly enough, we anticipate that we will need to restrict to a subcategory of abstract groups in order to find such a functor.

From http://www.arithgeo.ethz.ch/alpbach2012/Program-2012-3:

Pro-unipotent completion: Definition of Γ^{un} as a solution of the universal problem for group homomorphisms $\Gamma \to U(\mathbb{Q})$, where U is a pro-unipotent group over $\mathbb{Q}$ (in particular, Γ^{un} is a pro-unipotent group over $\mathbb{Q}$).

By Quillen, the Hopf algebra of regular functions on Γ^{un} is given by the formula

$$\mathcal{O}(\Gamma^{un}) = \varinjlim_n \left(\mathbb{Q}[\Gamma]/I^n\right)^\vee,$$

where $I \subset \mathbb{Q}[\Gamma]$ is the augmentation ideal and the product on the right hand side is dual to the map uniquely defined by $\gamma \mapsto 1 \otimes \gamma + \gamma \otimes 1$. In concrete way Γ^{un} may be given using the torsion free lower central series and explicit set of coordinates (this is probably the original construction of Malcev!). Correspondence between finite-dimensional representations of Γ^{un} and unipotent finite-dimensional representations of Γ, that is, representations of Γ that are extensions of trivial ones.

From Dalakov's "Relative Malcev Completion", http://math.mit.edu/conferences/talbot/2011/notes/talbot_2011_16_peter_notes.pdf:

Let π be a discrete group, R a commutative ring with 1 and $R\pi$ the group algebra

Let π be a discrete group, R a commutative ring with 1 and $R\pi$ the group algebra of π, that is, the set of all finite linear combinations $\sum_{g\in\pi} r_g g$, $r_g \in R$. Then the augmentation ideal J is the kernel of the ring homomorphism $\epsilon : R\pi \to R$, $g \mapsto 1$. The powers J^k determine a topology on $R\pi$ (often non-separated), and the J-adic completion is the topological R-algebra

$$\widehat{R\pi} = \lim_{\leftarrow} R\pi/J^k.$$