Kapovitch's class, Mon Jan 26: Rafi on Geometric Structures in 3D

January-26-15 11:05 AM

In 2D: S^2, $\mathbb{E}^2$, $\mathbb{H}^2$

In 3D:

$$\begin{tikzcd} & S^2 \arrow[dl, "\times"'] \arrow[dr, "\ltimes"] & \\ S^2\times S^1 & & S^3 \end{tikzcd}
\quad
\begin{tikzcd} & \mathbb{E}^2 \arrow[dl, "\times"'] \arrow[dr, "\ltimes"] & \\ \mathbb{E}^3 & & Nil \end{tikzcd}
\quad
\begin{tikzcd} & \mathbb{H}^2 \arrow[dl, "\times"'] \arrow[dr, "\ltimes"] & \\ \mathbb{H}^2\times\mathbb{R} & & \widetilde{SL(2,\mathbb{R})} \end{tikzcd}$$

(Hopf fibration)

Nil:

$$1 \longrightarrow \mathbb{R} \longrightarrow Nil \longrightarrow \mathbb{R}^2 \longrightarrow 0$$

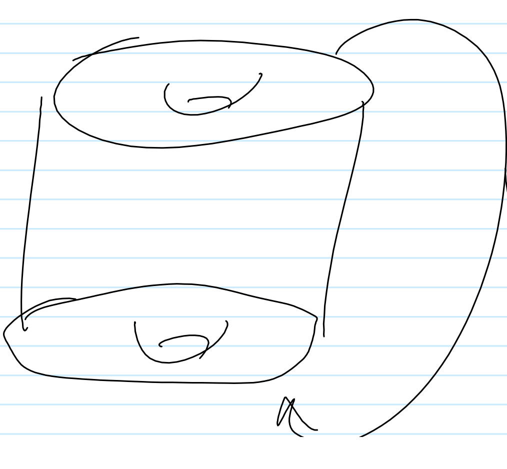
$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{metric: } dx^2+dy^2+(dz-y\,dx)^2$$

$\widetilde{SL(2,\mathbb{R})}$: Really, this is the geometry of the tangent space $T\mathbb{H}^2$.

Two further geometries:

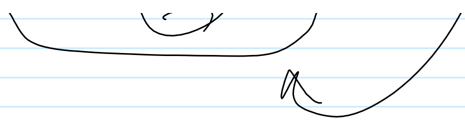
Sol: $0 \longrightarrow \mathbb{R}^2 \longrightarrow Sol \longrightarrow \mathbb{R} \longrightarrow 1$

Example



$\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$T^2\times I/(x,0)\sim$



$T^2 \times I / (x,0) \sim (\varphi(x),1)$

here φ is Anosov, conjugate to $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

$\mathbb{H}^3$ similar description as Sol, but starting from a higher-genus surface, using ψ-Anosov $\varphi: \Sigma_g \circlearrowleft$

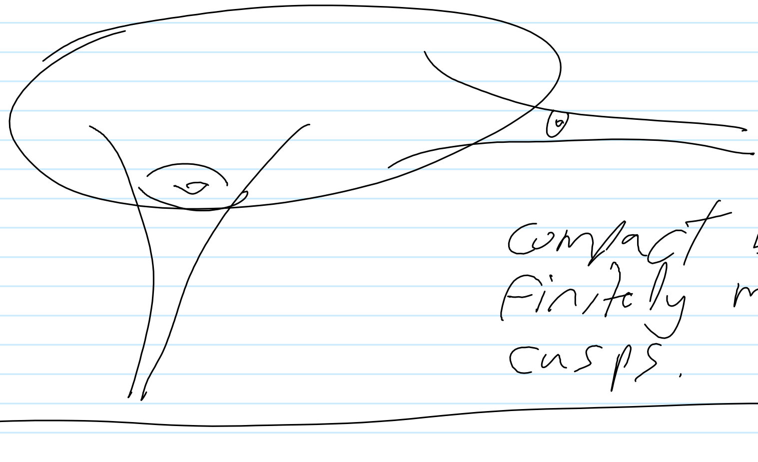
<u>The Virtual Fibered Theorem</u> (Last step by Agol)

Every closed / finite volume hyperbolic 3-manifold has a cover which is a surface bundle over a circle.

<u>Thm</u> Every geometry in 3-d with a co-compact lattice is one of the above.

... From a Theorem of Margulis:

Every finite volume 3-manifold has the shape:



compact body with finitely many $T \times \mathbb{R}$ cusps.

If M has a geometry then $\tilde{M}$ is $S^2\times\mathbb{R}$ or S^3 or $\mathbb{R}^3$

--- No essential spheres.

M is "prime" if it does not have essential spheres.

Theorem Every M^3 is a connected sum of primes, in a unique manner.

M is "atoroidal" if it does not contain essential tori.

Thm JSJ: For prime M w/ $\tilde{M}\not\cong S^2\times\mathbb{R}$, $\exists$ a decomposition along incompressible tori so that each piece is either

- Seifert fibered

or

- Atoroidal aspherical mfd with torus boundaries.

Thm The latter are always hyperbolic.